Response of a fractional oscillator to multiplicative trichotomous noise

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Abstract: We explore the phenomenon of stochastic resonance occurring in a fractional oscillator subjected to an external periodic force. The influence of fluctuations of environmental parameters on the dynamics of the oscillator is modeled as a multiplicative three-level Markovian noise. Using the Shapiro-Loginov formula, exact expressions for the response to an external periodic field are found. It is shown that there exists a critical memory exponent (a fractional exponent) which marks a dynamical transition in the behavior of the system. Particularly, it is demonstrated that the spectral amplification of the output signal exhibits a resonance-like nonmonotonic dependence on noise parameters. Influence of the memory exponent on friction-induced reentrant transitions between different resonance regimes of the oscillator is also discussed.

Key–Words: Stochastic oscillator, fractional oscillator, stochastic resonance, trichotomous noise, hypersensitive response, viscoelastic friction

1 Introduction

Active analytical and numerical studies of various dynamical models with random perturbations have been stimulated by their possible applications in different fields, ranging from ecosystems [1, 2] to intracellular protein transport in biology [3]–[5], or to methods of particle separation in nanotechnology [5, 6]. One of the objects of special attention in this context is the noise-driven harmonic oscillator [7]–[10]. Since non-linearity presents some difficulties for theoretical analysis of stochastic resonance phenomena, linear models of oscillators with multiplicative noise are of a particular interest. These models, on the one hand, show quasi-nonlinear behavior including stochastic resonance [10], and on the other hand, they allow exact analytical treatment.

A popular generalization of the harmonic oscillator, called the fractional oscillator, consists in the replacement of the usual friction term in the dynamical equation for a harmonic oscillator by a generalized friction term with a power-law type memory [11]–[14]. The main advantage of this equation is that it provides a physically transparent and mathematically tractable description of stochastic dynamics in systems with slow relaxation processes and with anomalously slow diffusion (subdiffusion). Examples of such systems are supercooled liquids, glasses, colloidal suspensions, dense polymer solutions [15, 16], viscoelastic media [17], and amorphous semiconductors [18]. Particularly, diffusion of mRNAs and ribosomes in the cytoplasm of living cells is anomalously slow [19], and large proteins behave similarly [20]. Even intrinsic conformational dynamics of protein macromolecules can be subdiffusive [21, 22].

In most model systems described as a fractional oscillator, the effect of a fluctuating environment on dynamical equations is taken into account as an additive white noise or fractional noise. However, it is well recognized that there are some important systems, especially in the context of biological applications, where the influence of a fluctuating environment should be modeled as a multiplicative colored noise, which has a non-zero correlation time [23]–[26]. Although the behavior of the fractional oscillator with an additive noise has been investigated in detail, it seems that analysis of the potential consequences of an interplay between multiplicative noise and memory effects is still missing in literature. This is quite surprising, because the importance of multiplicative fluctuations and viscoelasticity for biological systems, e.g., for living cells, has been well recognized [20, 26].

Thus motivated, we consider a fractional oscillator with a power-law memory kernel. The influence of the fluctuating environment is modeled by a multiplicative three-level Markovian noise (trichotomous noise). Although both dichotomous and trichotomous noises may be useful in modeling natural colored fluctuations, the latter is more flexible, including all cases of dichotomous noise [27, 28]. Furthermore, it is remarkable that for trichotomous noises the flatness parameter κ can be anything from 1 to ∞, unlike the flatness for Gaussian colored noise, $\kappa = 3$, and symmetric dichotomous noise, $\kappa = 1$. This extra degree of freedom can prove useful in modeling actual fluctuations.

The main contribution of this paper is as follows. We provide exact formulas for analytic treatment of the dependence of the mean oscillator displacement in the long-time limit, $t \to \infty$, on system parameters. On the basis of those exact expressions we will show that stochastic resonance (SR) is manifested in the dependence of the response of the noisy fractional oscillator upon the noise parameters, shuch as amplitude, correlation time, and flatness. To avoid misunderstanding, let us mention that we use the term SR in the wide sense, meaning nonmonotonic behavior of the output signal or some function of it, e.g., moments, in response to noise parameters [29]. Furthermore, we will show that at high values of noise flatness the output signal of the oscillator exhibits a hypersensitive response to noise amplitude. Moreover, we have found a critical memory exponent below which friction-induced reentrant transitions between different SR regimes of the oscillator appear.

The structure of the paper is as follows. A brief description of a trichotomous noise is presented in Section 2. In Section 3 we introduce the basic model investigated. Exact formulas for the mean oscillator displacement are derived. In Section 4 we analyze the behavior of the output response, and present the main results of this paper. Section 5 contains some brief concluding remarks.

2 Trichotomous noise

The trichotomous process $Z(t)$ [27] is a random stationary Markovian process that consists of jumps between three values a, 0, and $-a$. The jumps follow in time according to a Poisson process, while the values occur with the stationary probabilities

$$p_s(a) = p_s(-a) = q, p_s(0) = 1 - 2q, \tag{1}$$

with $0 < q \leq 1/2$. The mean value of $Z(t)$ and the correlation function are

$$\langle Z(t) \rangle = 0, \ \langle Z(t+\tau)Z(t) \rangle = 2qa^2 e^{-\nu\tau}. \tag{2}$$

It can be seen that the switching rate ν is the reciprocal of the noise correlation time τ_c, i.e, $\tau_c = 1/\nu$. The flatness parameter κ of the noise $Z(t)$ proves to be a very simple expression of the probability q

$$\kappa := \frac{\langle Z^4(t) \rangle}{\langle Z^2(t) \rangle^2} = \frac{1}{2q}. \tag{3}$$

The probabilities $W_n(t)$ that $Z(t)$ is in the state $n \in \{1, 2, 3\}$, $z_1 = a$, $z_2 = 0$, $z_3 = -a$, at the time t evolve according to the master equation

$$\frac{d}{dt} W_n(t) = \nu \sum_{m=1}^{3} S_{nm} W_m(t), \tag{4}$$

where

$$S_{nm} = \begin{pmatrix} q-1 & q & q \\ 1-2q & -2q & 1-2q \\ q & q & q-1 \end{pmatrix}. \tag{5}$$

The transition probabilities $T_{ij} = p(z_i, t+\tau | z_j, t)$ between the states z_n, $n = 1, 2, 3$, can be represented by means of the transition matrix T_{ij} of the trichotomous process as follows

$$T_{ij} = \delta_{ij} + (1 - e^{-\nu\tau}) S_{ij}, \tag{6}$$

where δ_{ij} is the Kronecker symbol. The trichotomous process is a particular case of the Kangaroo process [30]. It is remarkable that the results of the present paper can be interpreted in terms of cross-correlation intensity between two dichotomous noises. Namely, the trichotomous noise $Z(t)$ can be represented as the sum of two cross-correlated zero-mean symmetric dichotomous noises $Z_1(t)$ and $Z_2(t)$, i.e.,

$$Z(t) = Z_1(t) + Z_2(t).$$

The dichotomous noises $Z_1(t)$ and $Z_2(t)$ are characterized as follows: $z_1, z_2 \in \{(1/2)a, -(1/2)a\}$ with $\nu_1 = \nu_2 = \nu$ and the correlation function

$$\langle Z_i(t) Z_j(t') \rangle = \rho_{ij} \frac{a^2}{4} e^{-\nu|t-t'|}, \ \ i, j = 1, 2, \tag{7}$$

where $\rho_{ii} = 1$ and $\rho_{ij} = \rho \in (-1, 1)$ with $i \neq j$ is the cross-correlation intensity of the noises $Z_1(t)$ and $Z_2(t)$. In this case the probability $q = (1+\rho)/4$, whence it follows that the correlation coefficient ρ and the flatness κ of the trichotomous noise $Z(t)$ must be related as

$$\kappa = \frac{2}{1+\rho}. \tag{8}$$

It is obvious that the noise flatness $\kappa = 2$ corresponds to $\rho = 0$, i.e., to the case of two statistically independent dichotomous noises. Let us note that such a cross-correlation between dichotomous noises may result from either of the two following reasons: the two noises are either partly of the same origin or are influenced by the same factors. Notably, some cross-correlation-induced effects have earlier been considered in the context of ratchet models [31], [32], where it has also been suggested that cross-correlation between colored noises may provide some understanding as to why structurally very similar motor proteins with two heads, such as kinesin and dynein motor families, move in opposite directions on the microtubules despite sharing the same environment and experiencing the same periodicity, like with the conventional kinesin and ncd [33].

3 Model and the exact solution

As a model for an oscillatory system strongly coupled with a noisy environment, we consider a trichotomically perturbed oscillator with a power law type memory friction kernel

$$\ddot{X} + \gamma \frac{d^\alpha}{dt^\alpha} X + [\omega^2 + Z(t)]X = A_0 \sin(\Omega t), \quad (9)$$

where $\dot{X} \equiv dX/dt$, $X(t)$ is the oscillator displacement, γ is a friction constant, and the fractional Caputo derivative with the memory exponent (fractional exponent) $0 < \alpha < 1$ is defined as in [34],

$$\frac{d^\alpha X}{dt^\alpha} := \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\dot{X}(t')}{(t-t')^\alpha} dt', \quad (10)$$

where $\Gamma(y)$ is the gamma function. Fluctuations of the eigenfrequency ω are expressed as a trichotomous process $Z(t)$.

As in this work we will restrict ourselves to the behavior of the first moment of the oscillator displacement $\langle X(t) \rangle$, all results are also applicable in models where an additive noise $\xi(t)$, which is statistically independent from $Z(t)$ and has a zero mean, is included in the right side of Eq. (9). For example, depending on the physical situation, the noise $\xi(t)$ can be regarded either as an internal noise, in which case its stationary correlation satisfies Kubos's second fluctuation-dissipation theorem [35] expressed as

$$\langle \xi(t+\tau)\xi(t) \rangle = \frac{k_B T \gamma}{\Gamma(1-\alpha)\tau^\alpha} \quad (11)$$

(here k_B is the Boltzmann constant and T is the temperature of the heat bath), or as an external noise, in which case the driving noise $\xi(t)$ and the dissipation may have different origins and no fluctuation-dissipation relation holds.

To find the first moment of X we use the well-known Shapiro-Loginov procedure [36], which for a trichotomous noise $Z(t)$ yields

$$\frac{d}{dt} \langle Z\Phi \rangle = \left\langle Z \frac{d}{dt} \Phi \right\rangle - \nu \langle Z\Phi \rangle, \quad (12)$$

where Φ is an arbitrary functional of the process $Z(t)$. From Eqs. (9) and (12) we thus obtain an exact linear system of six first-order integro-differential equations for six variables: $x_1 = \langle X \rangle$, $x_2 = \langle \dot{X} \rangle$, $x_3 = \langle ZX \rangle$, $x_4 = \langle Z\dot{X} \rangle$, $x_5 = \langle Z^2 X \rangle$, $x_6 = \langle Z^2 \dot{X} \rangle$:

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = -\omega^2 x_1 - x_3 - \gamma \frac{d^\alpha}{dt^\alpha} x_1 + A_0 \sin(\Omega t),$$

$$\dot{x}_3 = -\nu x_3 + x_4,$$

$$\dot{x}_4 = -\nu x_4 - \omega^2 x_3 - x_5 - \gamma e^{-\nu t} \frac{d^\alpha}{dt^\alpha} \left(e^{\nu t} x_3 \right),$$

$$\dot{x}_5 = -\nu x_5 + x_6 + 2qa^2 \nu x_1,$$

$$\dot{x}_6 - 2qa^2 \dot{x}_2 = -\nu \left(x_6 - 2qa^2 x_2 \right) - a^2(1-2q)x_3$$

$$-\omega^2 \left(x_5 - 2qa^2 x_1 \right) - \gamma e^{-\nu t} \frac{d^\alpha}{dt^\alpha} \left[e^{\nu t} \left(x_5 - 2qa^2 x_1 \right) \right]. \quad (13)$$

The solution of equations (13) can be formally represented in the form

$$x_i(t) = \sum_{k=1}^{6} H_{ik}(t) x_k(0)$$

$$+A_0 \int_0^t \left[H_{k2}(t') + 2qa^2 H_{k6}(t') \right] \sin\left[\Omega\left(t - t'\right) \right] dt', \quad (14)$$

where the constants of integration $x_k(0)$ are determined by initial conditions. The relaxation functions $H_{ik}(t)$ with the initial conditions $H_{ik}(0) = \delta_{ik}$ can be obtained by means of the Laplace transformation technique. Particularly, we find that

$$\hat{h}(s) := \hat{H}_{12}(s) + 2qa^2 \hat{H}_{16}(s) = \frac{1}{D(s)}$$

$$\times \left\{ \left[(s+\nu)^2 + \gamma (s+\nu)^\alpha + \omega^2 \right]^2 - (1-2q)a^2 \right\}, \quad (15)$$

where

$$D(s) = (1-2q)a^2\left[\gamma(s+\nu)^\alpha - \gamma s^\alpha + \nu(2s+\nu)\right]$$
$$+\left[(s+\nu)^2+\gamma(s+\nu)^\alpha+\omega^2\right]\left\{\left(s^2+\gamma s^\alpha+\omega^2\right)\right.$$
$$\left.\times\left[(s+\nu)^2+\gamma(s+\nu)^\alpha+\omega^2\right]-a^2\right\}, \quad (16)$$

and $\widehat{H}_{ik}(s)$ is the Laplace transform of $H_{ik}(t)$, i.e.,

$$\widehat{H}_{ik}(s) = \int_0^\infty e^{-st} H_{ik}(t)dt. \quad (17)$$

One can check the stability of solution (14), which, according to the results of Ref. [37], means that the solutions s_j of the equation $D(s) = 0$ cannot have roots with a positive real part. This requirement is met if the inequality

$$a^2 < a_{cr}^2 = \frac{\omega^2(\omega^2+\nu^2+\gamma\nu^\alpha)^2}{[\omega^2+2q(\nu^2+\gamma\nu^\alpha)]} \quad (18)$$

holds. Henceforth in this work we shall assume that condition (18) is fulfilled. Thus in the long-time limit, $t\to\infty$, the memory about the initial conditions will vanish as

$$\sum_{k=1}^{6} H_{1k}(t)x_k(0) = \frac{\gamma\,\widehat{h}(0)\,x_1(0)}{\Gamma(1-\alpha)\,t^\alpha} + O\left(t^{-(1+\alpha)}\right) \quad (19)$$

and the average oscillator displacement $\langle X\rangle_{as} \equiv \langle X\rangle_{|t\to\infty}$ is given by

$$\langle X\rangle_{as} = A_0\int_0^t h(t-t')\sin(\Omega t')dt'. \quad (20)$$

From Eq. (20) it follows that the complex susceptibility $\chi(\Omega)$ of the dynamical system (9) is given by

$$\chi(\Omega) = \chi'(\Omega) + i\chi''(\Omega) = \widehat{h}(-i\Omega), \quad (21)$$

where $\chi'(\Omega)$ and $\chi''(\Omega)$ are the real and the imaginary parts of the susceptibility, respectively. Equation (20) can be written by means of the complex susceptibility as

$$\langle X\rangle_{as} = A\sin(\Omega t+\phi) \quad (22)$$

with the output amplitude

$$A = A_0\cdot|\chi| \quad (23)$$

and the phase shift

$$\phi = \arctan\left(-\frac{\chi''}{\chi'}\right). \quad (24)$$

Using Eqs. (15) and (16) we obtain for A that

$$A^2 = A_0^2\,\frac{C_1}{C_2}, \quad (25)$$

where

$$C_1 = \left[g_1^2+g_3^2-(1-2q)a^2\right]^2 + 4(1-2q)a^2g_3^2,$$

$$C_2 = \left[g_2^2+g_4^2\right]C_1 + 4qa^2\left\{\left(g_1^2+g_3^2\right)(g_3g_4-g_1g_2)\right.$$
$$\left.+a^2\left[q\left(g_1^2+g_3^2\right)+(1-2q)(g_1g_2+g_3g_4)\right]\right\} \quad (26)$$

and

$$g_1 = \omega^2+\nu^2-\Omega^2+\gamma(\nu^2+\Omega^2)^{\frac{\alpha}{2}}\cos\left[\alpha\,\arctan\left(\frac{\Omega}{\nu}\right)\right],$$

$$g_2 = \omega^2-\Omega^2+\gamma\Omega^\alpha\cos\left(\frac{\pi\alpha}{2}\right),$$

$$g_3 = 2\Omega\nu+\gamma(\nu^2+\Omega^2)^{\frac{\alpha}{2}}\sin\left[\alpha\,\arctan\left(\frac{\Omega}{\nu}\right)\right],$$

$$g_4 = \gamma\Omega^\alpha\sin\left(\frac{\pi\alpha}{2}\right). \quad (27)$$

The analytical expressions (25)–(27) belong to the main results of this work. They fully determine the behavior of the average oscillator displacement in response to system parameters in the long-time limit.

Finally, from Eqs. (15), (16), and (21) one can conclude that the real and the imaginary parts of the susceptibility are given by

$$\chi' = \frac{1}{C_2}\left[g_2C_1 - 2qa^2g_1\left(g_1^2+g_3^2-(1-2q)a^2\right)\right],$$

$$\chi'' = \frac{1}{C_2}\left[g_4C_1 + 2qa^2g_3\left(g_1^2+g_3^2+(1-2q)a^2\right)\right].$$

4 Stochastic resonance

Our next task is to examine the dependence of the response A on the noise amplitude a. To obtain some insight into the behavior of $\langle X\rangle_{as}$ at various system parameter regimes we first consider the adiabatic noise (i.e., $\nu\to 0$). At the long-correlation-time limit $\tau_c\to\infty$, the output amplitude A is given by

$$A^2 = \frac{A_0^2\left[(f_1-(1-2q)a^2)^2+f_2^2\right]}{f_3\left[(f_1-a^2)^2+f_2^2\right]}, \quad (28)$$

where

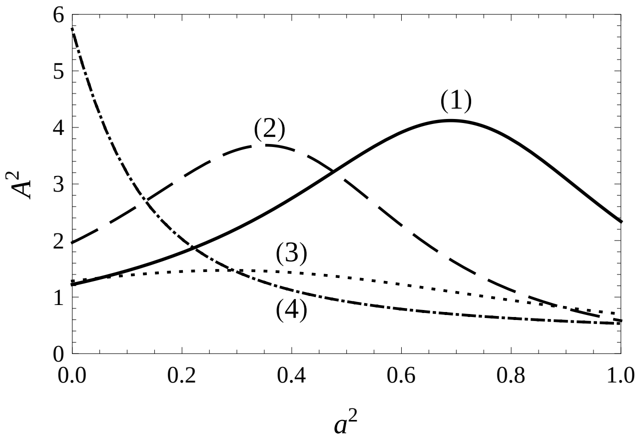


Fig. 1. SR for the response function A vs the noise amplitude a at various values of the friction coefficient γ. Other parameter values: $A_0 = \omega = 1$, $\alpha = 0.1$, $\nu = 0$, $q = 0.4$, and $\Omega = 1.8$. (1) Solid line: $\gamma = 1.3$; (2) dashed line: $\gamma = 1.5$; (3) dotted line: $\gamma = 2.85$; (4) dashed-dotted line: $\gamma = 2.3$. Note that in the case of curve (4), $\gamma = 2.3$, the phenomenon of stochastic resonance is absent.

$$f_1 = f_4^2 - \gamma^2\Omega^{2\alpha}\sin^2\left(\frac{\pi\alpha}{2}\right),$$

$$f_2 = 2\gamma\Omega^{\alpha} f_4 \sin\left(\frac{\pi\alpha}{2}\right),$$

$$f_3 = f_4^2 + \gamma^2\Omega^{2\alpha}\sin^2\left(\frac{\pi\alpha}{2}\right),$$

$$f_4 = \left[\omega^2 - \Omega^2 + \gamma\Omega^{\alpha}\cos\left(\frac{\pi\alpha}{2}\right)\right]. \quad (29)$$

In Fig. 1 we depict the behavior of $A(a)$ for various values of the system parameters. As is shown in Fig. 1, curves (1) - (3) exhibit a resonance-like maximum at some values of a, i.e., a typical SR phenomenon appears at increase of a. The existence of such an SR effect depends strongly on other system parameters. From Eqs. (28) and (29) one can easily find the necessary and sufficient conditions for the emergence of SR due to noise amplitude variations. Namely, nonmonotonic behavior of $A(a)$appears in the stability regions, $0 < a < a_{cr}$ (see Eq. (18)), for the parameter regime where the following inequalities hold:

$$2(1-q)f_3^2\omega^4 > f_1\left[f_3^2 + (1-2q)\omega^8\right] > 0. \quad (30)$$

In this case the response $A(a)$ reaches the maximum at

$$a_m^2 = \frac{f_3}{f_1(1-2q)}\left[(1-q)f_3 - \sqrt{q^2 f_3^2 + (1-2q)f_2^2}\right]. \quad (31)$$

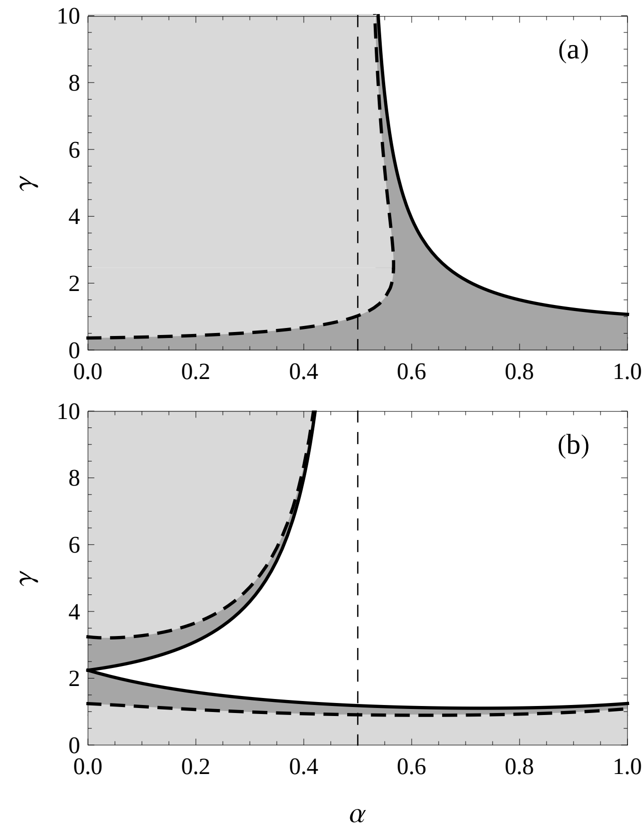


Fig. 2. A plot of the phase diagrams for SR in the γ – α plane at $A_0 = \omega = 1$, $q = 0.4$, and $\nu = 0$. In the unshaded region resonance of A vs the noise amplitude a is impossible. In the light grey region the function $A(a)$ exhibits a maximum at $a_m > \omega^2$, i.e., at a_m the first moment of the oscillator displacement $\langle X(t)\rangle$ is unstable, see Eq. (18). In the dark grey domain (the stability region) a stochastic resonance for A vs a occurs. The thin dashed line depicts the position of the critical memory exponent $\alpha_c = 1/2$. Panel (a): $\Omega = 0.6$; panel (b): $\Omega = 1.8$.

In Fig. 2 the conditions (30) are illustrated in the parameter space (γ, α) with two panels. The dark grey shaded domains in the figure correspond to those regions of the parameters γ and α, where SR versus a is possible. Note that in the light grey regions the response $A(a)$ formally also exhibits a resonance-like maximum, but in those regions the first moment $\langle X(t)\rangle$ is unstable at the resonance regime and that renders formula (28) physically meaningless. The boundaries $\gamma_{1,2}(\alpha)$ of the regions where SR vs a is possible are determined by the inequality $a_m^2 > 0$ with Eq. (31). From Eqs. (31) and (29) it follows that

$$\gamma_{1,2}(\alpha) = \frac{(\Omega^2 - \omega^2)}{\Omega^{\alpha}\cos(\pi\alpha)}\left[\cos\left(\frac{\pi\alpha}{2}\right) \pm \sin\left(\frac{\pi\alpha}{2}\right)\right].$$

Two findings can be pointed out. The first is the existence of a critical memory exponent $\alpha_c = 1/2$, which marks a sharp transition in the behavior of systems with fractional dynamics. At α_c, one of the boundaries $\gamma(\alpha)$ between the resonance and no-resonance regions tends to infinity.

The second finding is that depending on the driving frequency Ω, two different cases can be discerned. (i) For $\Omega^2 < \omega^2$, resonance vs a appears in the stability region for all values of γ when $\alpha < \alpha_c$, but if $\alpha > \alpha_c$, there is an upper border $\gamma(\alpha)$ above which the resonance is absent (Fig. 2(a)). (ii) In the case of $\Omega^2 > \omega^2$, if $\alpha < \alpha_c$, the interesting peculiarity of the diagram is that there are two disconnected regions (the shaded areas in Fig. 2(b)) where the resonance can appear. Thus, in this case a variation of the values of the friction parameter γ induces reentrant transitions between different dynamical regimes of the oscillator. Namely, an increase of γ can induce transitions from a regime where SR vs a is possible to the regime where SR is absent, but SR appears again through a reentrant transition at higher values of γ.

Next we consider the general case, $\nu \neq 0$ (see Eqs. (25)–(27)). In this case the regions in the parameter space $(\gamma - \alpha)$ where SR versus the noise amplitude a is possible are determined by the inequality $a_m^2 > 0$, where a_m^2 is the real solutions of the equation

$$x^2(1-2q)\left\{(1-2q)\left(g_1^2+g_3^2\right)(g_3g_4-g_1g_2) +2\left(g_1^2-g_3^2\right)\left[q\left(g_1^2+g_3^2\right)+(1-2q)(g_1g_2+g_3g_4)\right]\right\} -2x\left(g_1^2+g_3^2\right)^2\left[q\left(g_1^2+g_3^2\right)+(1-2q)(g_1g_2+g_3g_4)\right] +\left(g_1^2+g_3^2\right)^3(g_1g_2-g_3g_4)=0. \quad (32)$$

One readily sees from Eq. (32) that a positive solution exists if and only if the following inequality holds:

$$g_1g_2 > g_3g_4. \quad (33)$$

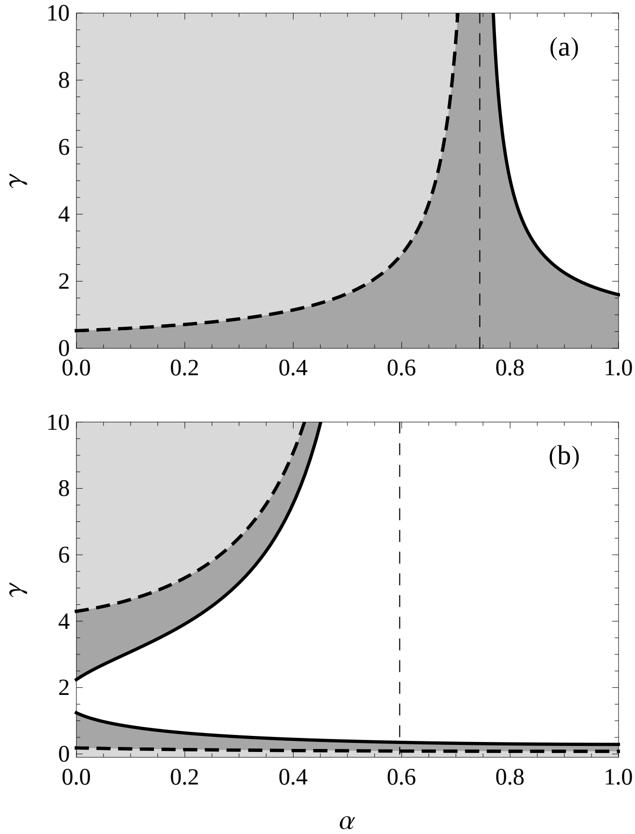
Thus the boundaries $\gamma_{1,2}(\alpha)$ of the resonance regions are given by $g_1g_2 = g_3g_4$. From Eqs. (27) it follows that

$$\gamma_{1,2} = \frac{1}{2d}\left(-b \pm \sqrt{b^2-4cd}\right), \quad (34)$$

where

$$c = \left(\omega^2-\Omega^2\right)\left(\omega^2+\nu^2-\Omega^2\right),$$

$$b = \Omega^\alpha\left(\omega^2+\nu^2-\Omega^2\right)\cos\left(\frac{\pi\alpha}{2}\right)-2\nu\Omega^{\alpha+1}\sin\left(\frac{\pi\alpha}{2}\right) +\left(\omega^2-\Omega^2\right)\left(\nu+\Omega^2\right)^{\frac{\alpha}{2}}\cos\left[\alpha\arctan\left(\frac{\Omega}{\nu}\right)\right],$$



Fiq. 3. The phase diagrams for SR vs a in the $\gamma - \alpha$ plane at $A_0 = \omega = 1$, $\nu = 1.0$, and $q = 0.5$. Panel (a): $\Omega = 0.6$; panel (b): $\Omega = 1.8$. In the unshaded region SR versus the noise amplitude a is impossible. In the light grey region the function $A(a)$ exhibits a maximum at $a_m > a_{cr}$; see Eq. (18). In the dark grey domain SR vs a occurs (in the stability region, $a_m < a_{cr}$). The thin dashed line depicts the position of the critical memory exponent a_c.

$$d = \Omega^\alpha(\nu^2+\Omega^2)^{\frac{\alpha}{2}}\cos\left[\alpha\left(\arctan\left(\frac{\Omega}{\nu}\right)+\frac{\pi}{2}\right)\right]. \quad (35)$$

It can be seen from Eqs. (34) and (35), that one of the boundaries $\gamma_{1,2}(\alpha)$ tends to infinity if d tends to zero. This happens when the memory exponent α tends to α_c, where

$$a_c = \frac{\pi}{\pi+2\arctan\left(\frac{\Omega}{\nu}\right)}. \quad (36)$$

Note that neither the phase boundaries $\gamma_{1,2}(\alpha)$ nor the critical memory exponent a_c depend on the noise parameter q. So in the general case the critical memory exponent a_c, which marks a sharp transition in the behavior of systems with fractional dynamics, depends only on the ratio of the driving frequency Ω to the noise switching rate ν (see also Fig. 3). Particularly, in the adiabatic case ($\nu \to 0$) the critical exponent a_c

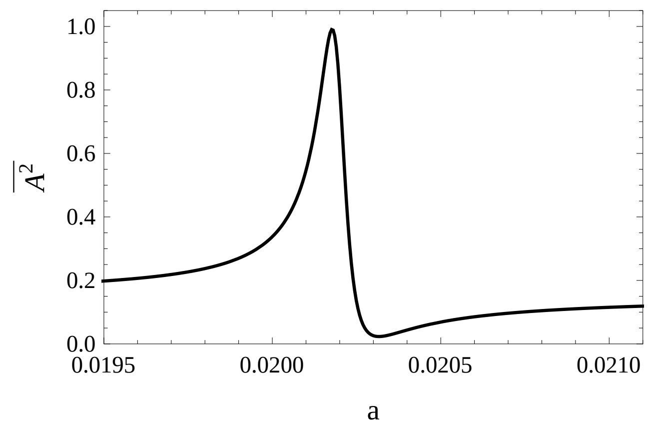


Fig. 4. A plot of the dependence of the response function A on the noise amplitude a in a region of hypersensitive response [Eqs. (28) and (37)]. System parameter values: $\gamma = 3 \cdot 10^{-4}$, $q = 5 \cdot 10^{-3}$, $\Omega = 0.99$, $\alpha = 0.1$, $\nu = 0$, and $A_0 = \omega = 1$. The value of A^2 at the local maximum is $A_m^2 = 15857$; $\overline{A^2} \equiv A^2/A_m^2$.

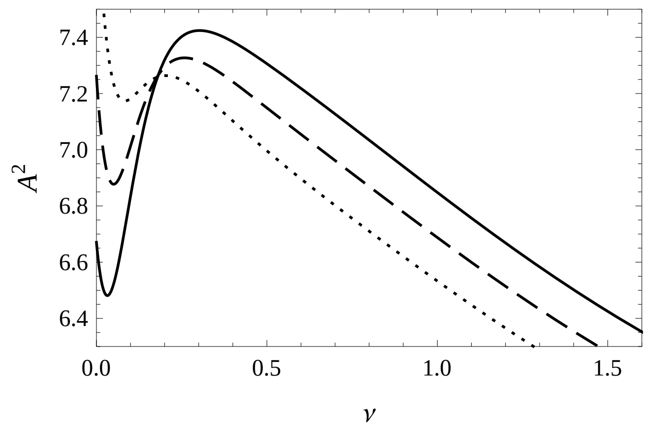


Fig. 5. SR for A^2 versus the noise switching rate ν, computed from Eq. (25)–(27) at various values of the friction coefficient γ. Other parameter values: $A_0 = \omega = 1$, $a^2 = 0.3$, $\alpha = 0.3$, $\Omega = 0.8$, and $q = 0.4$. Solid line: $\gamma = 0.09$; dashed line: $\gamma = 0.095$; dotted line: $\gamma = 0.1$.

is 0.5 and in the fast-noise limit ($\nu \to \infty$) a_c tends to 1. From analysis of Eqs. (33)-(35) it follows that depending on the driving frequency Ω, three different regimes of the dynamical system (9) can be discerned.(i) For $\Omega^2 < \omega^2$, SR vs a appears in the stability region for all values of γ when $\alpha < \alpha_c$, but if $\alpha > \alpha_c$, there is an upper border $\gamma(\alpha)$ above which SR is absent (Figs. 2(a), 3(a)). (ii) In the case of $\omega^2 < \Omega^2 < \omega^2 + \nu^2$ for $\alpha < \alpha_c$ the resonance exists only if $\gamma > \Omega^2 - \omega^2$; in the region $\alpha > \alpha_c$ the resonance is absent. (iii) At the driving frequency regime $\Omega^2 > \omega^2 + \nu^2$, if $\alpha < \alpha_c$, there are two disconnected regions (Figs. 2(b) and 3(b)) where SR vs a is possible. An important observation here is that the region where the resonance is not possible grows as the noise switching rate ν increases (cf. Figs. 2(b) and 3(b)). This tendency is in accordance with the fact that at high values of the noise switching rate the system (9) behaves as a deterministic fractional oscillator (without noise).

Finally we consider, in brief, another interesting SR phenomenon – hypersensitive response to noise amplitude. A peculiarity of Fig. 4 is a rapid decrease of A^2 from maximum to minimum as a increases. It is noteworthy that in the case of dichotomous noise such an effect is absent. The effect is very pronounced at low values of the damping constant γ. To throw some light on the above-mentioned effect, we shall now briefly consider the behavior of the SR characteristic A^2 in the parameter regime $\nu = 0$, and

$$\gamma\Omega^\alpha << q\left|\omega^2 - \Omega^2\right| << \omega^2,\ q << 1. \tag{37}$$

In this case, it follows from Eqs. (28) and (29) that A^2 reaches the maximum

$$A_{max}^2 \approx A_0^2 \frac{q^2}{\gamma^2\Omega^{2\alpha}} \tag{38}$$

at

$$a = a_{max} \approx \left|\omega^2 - \Omega^2\right|, \tag{39}$$

and the minimum

$$A_{min}^2 \approx A_0^2 \frac{\gamma^2\Omega^{2\alpha}}{q^2\left(\omega^2 - \Omega^2\right)^4} \tag{40}$$

at

$$a = a_{min} \approx \frac{\left|\Omega^2 - \omega^2\right|}{\sqrt{1-2q}}. \tag{41}$$

For sufficiently strong inequalities (37), A_{min}^2 tends to zero and A_{max}^2 grows up to very large values. Thus in the case considered the response A is extremely sensitive to a small variation of a: $\Delta a = a_{min} - a_{max} \approx q\left|\Omega^2 - \omega^2\right|$.

Formulas (37) – (41) are exactly the same as can be derived from the results of [38] for an ordinary stochastic oscillator (without memory, $\alpha = 1$) if we replace the friction coefficient γ with the corresponding effective quantity $\gamma_{ef} = \gamma\Omega^{\alpha-1}$. This suggests that the physical explanation of the effect of hypersensitive response to noise amplitude exposed in [38] for a stochastic oscillator without memory is also applicable in the case of a stochastic fractional oscillator.

The phenomenon of SR is not restricted to nonmonotonic dependence of A on the noise amplitude a. Figures 5 and 6 depict the behavior of the response A versus the noise switching rate ν and versus the noise flatness parameter $q = 1/(2\kappa)$, respectively. In Figure

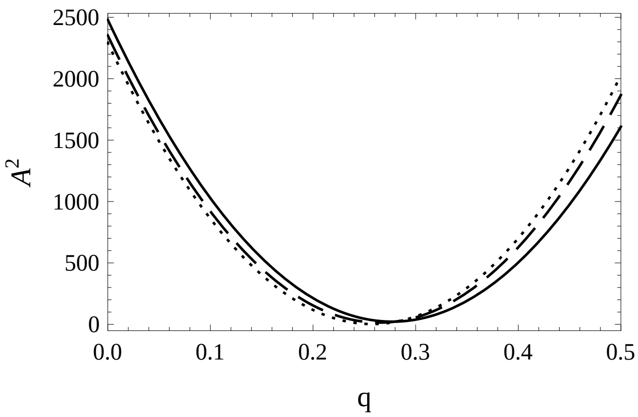


Fig. 6. Dependence of the SR characteristic A^2 on the noise parameter q in the case of a long correlation time. The curves correspond to the following parameters: $A_0 = \omega = 1$, $a = 0.03$, $\Omega = 0.99$, $\gamma = 0.001$, and $\nu = 0.0001$; (Eqs. (25)–(27)). Solid line: $\alpha = 0.9$; dashed line: $\alpha = 0.5$; dotted line: $\alpha = 0.1$.

5, one observes resonance versus ν, which apparently gets more and more pronounced as the friction coefficient γ decreases. It is remarkable that in contrast to the case A vs a, SR vs ν depends on the memory exponent α very weakly: as α increases from zero to 1 only a slight deformation of the curves $A(\nu)$ can be observed.

A plot (Fig. 6) of the response A^2 versus the noise parameter q for different values of the memory exponent α shows a typical SR with nonmonotonic behavior of the function $A^2(q)$. At a long correlation time, $\nu \to 0$, which is the case considered in Fig. 6, it follows from Eq. (28) that on condition

$$a^2 > f_1 > 0 \tag{42}$$

the response A^2 reaches a minimum at

$$q = q_m = \frac{1}{2a^2}\left(a^2 - f_1\right). \tag{43}$$

Note that the inequalities (42) are the necessary and sufficient conditions for the SR phenomenon vs q to occur in the adiabatic limit. Evidently, if the damping parameter γ is low, the suppression of A^2 at $q = q_m$ is very pronounced, i.e., $A^2(q_m)$ tends to zero as γ vanishes. It is seen from Fig. 6 that the influence of the memory exponent α on the resonance behavior of $A^2(q)$ is very weak.

5 Conclusions

In the present work, we have analysed the phenomenon of stochastic resonance within the context of a noisy, fractional oscillator with a fluctuating eigenfrequency driven by a sinusoidal forcing. The viscoelastic type friction kernel with memory is assumed as a power-law function of time and the eigenfrequency fluctuations are modeled as a colored three-level Markovian noise. The Shapiro-Loginov formula [36] with the Laplace transformation technique allow us to find an exact expression for the long-time behavior of the mean oscillator displacement.

As one of the main results we have established the effect of a very sensitive response of the mean oscillator displacement to small variations of the noise amplitude at high values of noise flatness, i.e., the amplitude of the output signal displays a quick jump from a very high value to a low one as the noise amplitude increases but a little. It is important to note that such a phenomenon has been previously reported for a stochastic oscillator without memory in Ref. [38]. As another main result we have found, first, the existence of a band gap for the values of the friction coefficient γ between two regions of the $(\gamma - \alpha)$ phase diagrams where SR vs noise amplitude is possible at sufficiently small values of the memory exponent $\alpha < \alpha_c(\Omega/\nu)$, and, second, the corresponding friction-induced reentrant transitions between these different dynamical regimes of the oscillator.

We believe that the results obtained are of interest also in cellbiology, where issues of memory and multiplicative colored noise can be crucial [12, 20, 22, 26].

A further detailed study is, however, necessary – especially an investigation of the behavior of second moments [39].

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