The Relationships Between the Diameter Growth and Distribution Laws

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Abstract - The processes of growth play an important role in different fields of science, such as biology, medicine, forestry, ecology, economics. Usually, in applied sciences the averaged trend kinetics is represented by means of logistic laws (Verhulst, Gompertz, Mitscherlich, von Bertalanffy, Richards etc.). We used a generalized stochastic logistic model for predicting the tree diameter distribution of forestry stands. The purpose of this paper was to develop a diameter probability density function for even-aged and uneven-aged stands using the stochastic logistic law of diameter's growth. The parameters of stochastic logistic growth law were estimated by the maximum likelihood procedure using a large dataset on permanent sample plots provided by Lithuanian National Forest Inventory. Subsequently, we numerically simulated the probability density function of diameter distribution for the Verhulst, Gompertz, Mitscherlich, von Bertalanffy, Richards stochastic growth laws. The exact solution (transition probability density function of diameter size) of the Fokker-Planck equation (the partial differential equation for evolving distribution of diameter size) was derived exclusively for the Gompertz stochastic growth law. The comparison of the goodness of fit among probability density functions was made by the normal probability plot and the *p*-value of the Kolmogorov-Smirnov and Cramer - von Mises tests. To model the diameter distribution, as an illustrative experience, is used a real data set from repeated measurements on permanent sample plots of pine stands in Dubrava district. The results are implemented in the symbolic computational language MAPLE.

Key-Words: - Diameter distribution, Stochastic differential equation, Density function, Fokker-Plank equation, Numerical solution.

1 Introduction

The processes of growth play an important role in various applied areas, such as biology, medicine, biochemical industry. The environment of any real system is in general not constant but shows random fluctuations. Despite this, the growth model historically has crystallized as the deterministic process [12], [22], [59], [60], [61].

Over the last few decades interest in the effects of noise in dynamical systems have significantly increased. Previous studies demonstrate different ways to incorporate stochastic structure into deterministic models [6], [47]. Though Gaussian white noise is very irregular, it is useful to model the randomness phenomenon. Regularly, Gaussian white noise is used to determine the stochastic influence into growth kinetics. Naturally true white noise does not occur in nature. When a deterministic system is affected by Gaussian white noise, it is often modeled by a stochastic differential equation. Stochastic models with Gaussian white noise find numerous applications in a variety of branches of science and technology. Evolution equations, driven by the white noise, play an important role in biology [33], medicine [6], [24], [32], [45], forestry [16], [49], economics [43], [66], [68]. The time-delayed stochastic differential equations demonstrate more complicated dynamics than ordinary differential equations [14], [50].

In general, stochastic processes can be characterized by means of transition probability density. Unfortunately, stochastic differential equations for which analytical results are known are very scarce. The main part of this paper is concerned with applications of the stochastic differential equations to the modeling of diameter growth and distribution of forest stands.

The kinetics of the stands under investigation plays an important role observing the processes in forestry. The processes of growth of forestry stands are usually described by means of an ordinary differential equation [11], [18], [19], [28], [37], [51], [70]. Conventionally, the mean trend kinetics is expressible by means of one of the types of logistic forms e.i. Verhulst, Gompertz, Mitscherlich, von Bertalanffy, Richards or their mixture [21], [46], [63], [64].

Diameter dynamics is affected by many processes and varies among stands [5], [53], [58]. Stochastic growth models allow us to reduce the unexplained variability of simulated growth variable and to implement the randomness phenomenon, which makes a stochastic influence on any process in practical applications. There are two types of approaches for this purpose [34]. The first approach is based on 'environment' stochasticity, introducing a diffusion term in the ordinary differential equation [9], [16]-[18], [49], [66]. The second approach is based on 'demographic' stochasticity in which the population size X is a random variable [2], [4], [13] [34]. In this paper we follow the first approach. If the diffusion coefficient is independent of state variable, the noise is called additive, otherwise it is multiplicative. The first studies within stochastic logistic growth model, where the fluctuations of growth dynamics is modeled by the additive (noise amplitude) random perturbations have been paper by Garcia [16] and both additive and multiplicative (state dependent) random perturbations by Willassen [66].

In the last 20 years the studies of forest management have relied on the discrete and continuous time stochastic growth models [1], [8]-[9], [43], [66], [68], [71]-[72]. Studies have predominantly focused on the impact of stochastic forest stand value and prices on the rotation age.

The main purpose of our study is to develop the age or the height dependent probability density function on diameter size using measurements of tree variables such as age, height, diameter. This paper not only provides useful stochastic models for the diameter growth modeling, but shows that it is possible to relate the diameter growth models and the diameter distribution models. The distributions of diameter size in stands describe forest structure and can be used for the assessment of stand volume and biomass [7], forest biodiversity and density management [39].

In even-aged stands various distribution functions, such as negative exponential, Pearson, gamma, lognormal, beta, Weibull, Johnson, Gram-Charlier, have been used in describing the diameter distributions [7], [23], [31], [35], [38], [40]-[42], [56], [65]. In uneven-aged stands have been used bivariate distributions and density mixtures [3], [27], [29]-[30], [49], [55], [62], [69], [73]. Our method combines the parameter prediction and parameter recovery approaches [35], [39].

In practical applications it has the advantage that the same family of distribution functions can be used throughout the whole life of stands' development and the parameters of growth model are related to stand-level characteristics such as dominant height, basal area, site index, stand density. In this paper the diameter distribution is analyzed as a mixture of distributions of trees belonging to various stages of stand development.

This article is designed to give readers a comprehensive and practical method to the numerical simulation of transition probability density functions, mean and variance trajectories of five stochastic logistic growth laws with a minimum of technical detail. Next, we have discussed the connection between stochastic logistic growth laws and certain time-dependent partial differential equations (Fokker-Planck equations).

2 Materials and Methods

2.1 Data

The diameter analysis is biased on experiments in Pine stands at Dubrava district in Lithuania. The data were provided by the Lithuanian National Forest Inventory, which consisted of a systematic sample plots distributed on a square grid of 5 km, with a 5 year remeasurement interval. The sample method used circular plots of fixed radius with the area 500m^2 [26]. The data analyzed were collected during 1976. During this time these stands have been remeasured 5 times on stand variables: age, number of trees per hectare, breast height diameter, trees position co-ordinates, age and height. The measurements have been conducted in 34 permanent treatment plots and the initial planting densities are unknown. The age of stands ranges from 12 to 103 years. The mean of diameter at breast height varies from 2.2 to 51.5 cm. Approximately 30% of sample trees in all plots are randomly selected for the height measurement. The observed data of study plots are presented in Figure 1.

2.2 Methods

In the present paper we study the dynamic behavior of tree diameter (diameter at breast height) and its relationship with diameter distribution law. The deterministic logistic model of tree diameter growth expressing the trajectory $x(t)$ of the process depending on time t (or other variable, ν, such as the height in meters) in its general form can be expressed by means of the ordinary differential equation [61]

$$\frac{dx(t)}{dt} = r(x(t))^{\alpha}\left(1-\left(\frac{x(t)}{K}\right)^{\beta}\right), \qquad (1)$$

$$x(t_0) = x_0,\ t \in [t_0, T],$$

where r, K, α, β are real numbers. The instantaneous diameter x grows, in the absence of any restraints, exponentially fast in time t with growth rate per individual equal to r. The actual evolution of the diameter is cut back by the saturation including term $\left(1-\left(\frac{x(t)}{K}\right)^{\beta}\right)$. The parameter K is called the carrying capacity of the environment and commonly represents the maximum diameter that can be supported by the resources of the environment. It is obvious that when the time approaches infinity, the trajectory $x(t)$ satisfying equation (1) approaches the carrying capacity (saturation level) K, that is, $\lim_{t\to\infty} x(t) = K$. The equilibrium solution x^* can be obtained by setting $x' = 0$. Hence, the single positive root is $x^* = K$. Some cases of the generalized logistic growth model (1) and their corresponding trajectories are presented in Table 1. The trajectories $x(t)$ of these laws depend exclusively on time t and parameters r, K, β. It should be noted, that the deterministic growth laws expressed by (1) cause high variation which cannot be explained by the model itself.

In this paper a generalized stochastic logistic diameter growth model is represented by an ordinary stochastic differential equation with both the additive and multiplicative noises in the following form

$$dX(t) = r(X(t))^{\alpha}\left(1-\left(\frac{X(t)}{K}\right)^{\beta}\right)dt + \sigma\begin{cases} dW(t) \\ X(t)dW(t) \end{cases},$$

$$X(t_0) = x_0 \qquad (2)$$

or in integrated form as

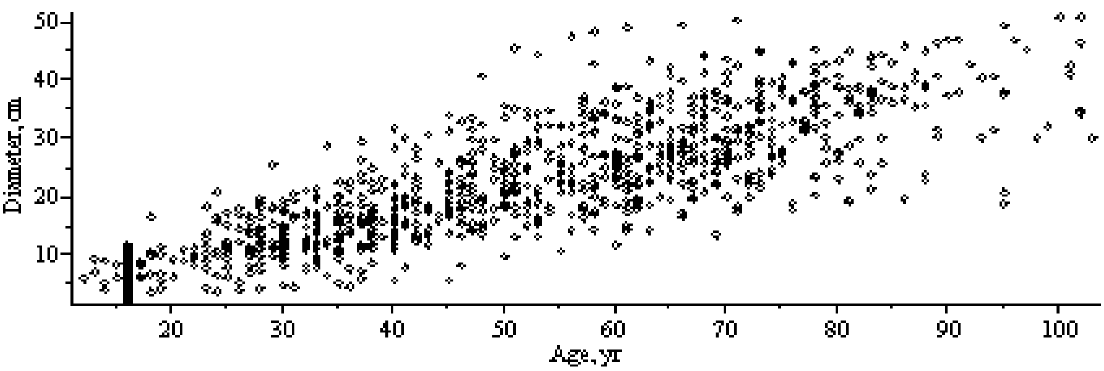
$$X(t) = X(t_0) + \int_{t_0}^{t} r(X(s))^{\alpha}\left(1-\left(\frac{X(s)}{K}\right)^{\beta}\right)ds + \int_{t_0}^{t}\sigma\begin{cases} dW(s) \\ X(s)dW(s) \end{cases},$$

$$t \in [t_0, T],$$

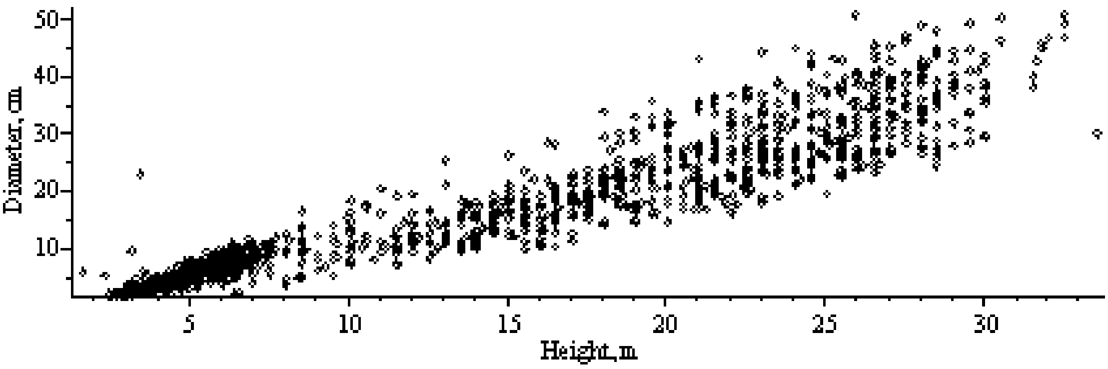
where $X(t)$ is the tree breast height diameter at the age t (or other variable, ν, such as the height in meters), σ represents the effect of the noise on the dynamics of diameter, $W(t)$ is a scalar Brownian motion, which is a random process whose increments are independent and normally distributed with zero mean and with variance equal to the length of the time interval over which the increment take place, x_0 is not random.

Here we have assumed the Ito interpretation of stochastic differential equation which does not conserve the ordinary rule of calculus.

The exact dynamics of the diffusion growth process (2) are governed by its transition probability density.



a)



b)

Fig. 1. Plot of the diameter including data from pine forests in Lithuania: a) the age dependent, b) the height dependent.

Table1. Trajectories of deterministic logistic growth laws.

Law	Parameters α	Parameters β	Trajectory $x(t)$
Verhulst	1	1	$\frac{Kx_0}{x_0+(K-x_0)e^{-r(t-t_0)}}$
Gompertz	1	$\beta \to 0$	$K\left(\frac{x_0}{K}\right)^{e^{r(t-t_0)}}$
Mitscherlich	0	1	$K+(x_0-K)e^{-\frac{r}{K}(t-t_0)}$
Bertalanffy	2/3	1/3	$K\left(1-\left(1-\left(\frac{x_0}{K}\right)^{\frac{1}{3}}\right)e^{-\frac{r(t-t_0)}{3K^{\frac{1}{3}}}}\right)^3$
Richards	1	$\beta \geq -1$	$\left(\frac{\left(\frac{x_0}{K}\right)^{\beta}e^{r\beta(t-t_0)}-\left(\frac{x_0}{K}\right)^{\beta}+1}{\left(\frac{x_0}{K}\right)^{\beta}}\right)^{-\frac{1}{\beta}}Ke^{r(t-t_0)}$

$$p(x,v)=P\left(X(v)\in dw \middle| X(v_0)=x_0;r,K,\alpha,\beta\right)/dw.$$

Unfortunately, in all except a Gompertz case the transition probability density are not analytically available.

The application of the stochastic model (2) requires to estimate the parameters r , K , β , σ of the drift term $rX(v)^{\alpha}\left(1-\left(\frac{X(v)}{K}\right)^{\beta}\right)$ and the diffusion term $\sigma(X(v))$ ($\sigma(X(v))=\sigma$ for the additive noise, $\sigma(X(v))=\sigma X(v)$ for the multiplicative noise). In order to obtain the evaluation of the above-mentioned parameters of the stochastic differential equation (2) using an observed data set it is possible to apply the maximum likelihood or L^1 norm procedures [48], [54]. An excellent review of inference for discretely observed diffusion processes is given in [57]. The approach that is used in this paper follows the maximum likelihood procedure, which exploits a local linearization methodology for diffusion processes. If parameters are estimated, we can further obtain the probability density function $p(x,v)$ of diameter distribution, which depends on the stand-level characteristics, v , such as age, height, basal area, stand density, site index of stands. Suppose, that

$$f\left(X(v)\right) = rX(v)^{\alpha}\left(1-\left(\frac{X(v)}{K}\right)^{\beta}\right),$$

where v can be the age t or the height h of tree or other predictor variable (basal area, stand density, site index). The probability density function $p(x,v)$ satisfies the Fokker-Plank partial differential equation (also known as the Chapman-Kolmogorov equation) [20]

$$\frac{\partial p(x,v)}{\partial v} = -\frac{\partial}{\partial x}\left(f(x)p(x,v)\right) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left((\sigma(x))^2 p(x,v)\right). \quad (3)$$

This equation models the evolution of physical an biological systems, where a main process is complicated by noise in the system.

The exact steady-state solutions $p(x)$ of equation (3) with multiplicative noise are represented in Table 2. These steady-state probability density functions have explicit parameters denominating the diameter intrinsic growth, the diameter carrying capacity and the intensity of noise. A simple approximation of the steady-state density function for the Verhulst type stochastic population growth model was discussed by Matis [34].

The practical application of the stochastic diameter growth models (2) needs to derive the first two moments, namely, the mean m_ν and the variance s_ν of diameter size. Ordinary differential equations for the first two moments ($m_\nu = E(X(\nu))$, $s_\nu = V(X(\nu))$) can be derived from the Fokker-Plank equation (3) or directly from the stochastic differential equation (2). Thus multiplying equation (3) with x and integrating it with respect to x, we get the evolution of mean $m_\nu = E(X(\nu))$. Likewise, multiplying equation (3) with x^2 and integrating it with respect to x, we get the evolution of variance $s_\nu = V(X(\nu))$. These equations describe the behavior over predictor variable, ν, the mean and variance of diameter size and, for the multiplicative noise, take the following form

$$\begin{cases} \dfrac{dm_\nu}{d\nu} = r\left(m_\nu - \dfrac{m_\nu^2 + s_\nu}{K}\right), \\ \dfrac{ds_\nu}{d\nu} = 2r\left(1-\dfrac{2m_\nu}{K}\right)s_\nu + \sigma^2\left(m_\nu^2 + s_\nu\right), \end{cases} \quad (4)$$

for the stochastic Verhulst growth law,

$$\begin{cases} \dfrac{dm_\nu}{d\nu} = rm_\nu \ln\dfrac{K}{m_\nu} - \dfrac{rs_\nu}{2m_\nu}, \\ \dfrac{ds_\nu}{d\nu} = 2rs_\nu\left(\ln\dfrac{K}{m_\nu} - 2\right) + \sigma^2\left(m_\nu^2 + s_\nu\right), \end{cases} \quad (5)$$

for the stochastic Gompertz growth law,

Table 2. Steady-state solutions of the Fokker-Plank equation (3).

Law	Steady-state density function	Normalizing constant N_s
Verhulst	$N_s x^{2\left(\frac{r}{\sigma^2}-1\right)} e^{-2\frac{rx}{K\sigma^2}}$	$\left(\frac{2r}{K\sigma^2}\right)^{\frac{2r}{\sigma^2}-1} / \Gamma\left(\frac{2r}{\sigma^2}-1\right)$
Gompertz	$N_s x^{-2} e^{-\frac{r\ln^2\frac{K}{x}}{\sigma^2}}$	$\frac{K}{\sigma}\sqrt{\frac{r}{\pi}} e^{-\frac{\sigma^2}{4r}}$
Mitscherlich	$N_s x^{-2\left(\frac{r}{K\sigma^2}+1\right)} e^{-2\frac{r}{\sigma^2 x}}$	$\left(\frac{2r}{\sigma^2}\right)^{\frac{2r}{K\sigma^2}+1} / \Gamma\left(\frac{2r}{K\sigma^2}+1\right)$
Bertalanffy	$N_s x^{-2\left(\frac{r}{K^{\frac{1}{3}}\sigma^2}+1\right)} e^{-6\frac{r}{x^{\frac{1}{3}}\sigma^2}}$	$\frac{1}{3}\left(\frac{6r}{\sigma^2}\right)^{3\left(\frac{2r}{K^{\frac{1}{3}}\sigma^2}+1\right)} / \Gamma\left(3\left(\frac{2r}{K^{\frac{1}{3}}\sigma^2}+1\right)\right)$
Richards	$N_s \left(\frac{x}{K}\right)^{2\left(\frac{r}{\sigma^2}-1\right)} e^{-2r\frac{\left(\frac{x}{K}\right)^\beta}{\beta\sigma^2}}$	$\frac{\beta}{K}\left(\frac{2r}{\beta\sigma^2}\right)^{\frac{1}{\beta}\left(\frac{2r}{\sigma^2}-1\right)} / \Gamma\left(\frac{1}{\beta}\left(\frac{2r}{\sigma^2}-1\right)\right)$

$$\begin{cases} \dfrac{dm_\nu}{d\nu} = r\left(1 - \dfrac{m_\nu}{K}\right) \\ \dfrac{ds_\nu}{d\nu} = -2\dfrac{r}{K}s_\nu + \sigma^2\left(m_\nu^2 + s_\nu\right), \end{cases} \quad (6)$$

for the stochastic Mitscherlich growth law,

$$\begin{cases} \dfrac{dm_\nu}{d\nu} = r\left(m_\nu^{\frac{2}{3}} - \dfrac{m_\nu}{K^{\frac{1}{3}}} - \dfrac{s_\nu}{9m_\nu^{\frac{4}{3}}}\right), \\ \dfrac{ds_\nu}{d\nu} = 2r\left(\dfrac{2}{3m_\nu^{\frac{1}{3}}} - \dfrac{1}{K^{\frac{1}{3}}}\right)s_\nu + \sigma^2\left(m_\nu^2 + s_\nu\right), \end{cases} \quad (7)$$

for the stochastic von Bertalanffy growth law and

$$\begin{cases} \dfrac{dm_\nu}{d\nu} = r\left(m_\nu\left(1 - \left(\dfrac{m_\nu}{K}\right)^\beta\right) - \beta(1+\beta)\dfrac{m_\nu^{\beta-1}s_\nu}{2K^\beta}\right), \\ \dfrac{ds_\nu}{d\nu} = 2r\left(1 - (1+\beta)\left(\dfrac{m_\nu}{K}\right)^\beta\right)s_\nu + \sigma^2\left(m_\nu^2 + s_\nu\right), \end{cases} \quad (8)$$

for the stochastic Richards growth law.

In the case of Gompertz law, the exact solution of the Fokker-Plank equation (3) has the following form [49]

$$p(x,\nu) = \frac{1}{\sigma x\sqrt{\pi\left(1 - e^{-2r\nu}\right)/r}} \times e^{-\frac{r\left(\ln x - \ln K + \sigma^2/2r - e^{-r\nu}\ln x_0\right)^2}{\sigma^2\left(1 - e^{-2r\nu}\right)}}. \quad (9)$$

This probability density function of tree diameter size depends on the predictor variable ν. The predictor variable ν can be any stand-level characteristics such as age, height, basal area, stand density, site index.

For all the above-mentioned stochastic logistic growth forms (2) many numerical methods for the solution of equation (3) can be used [44]. The numerical solution to the initial-boundary value problem (3) is computed using the Crank-Nicolson finite difference scheme, whose discretization error is of second order in both space and time. This scheme ensures numerical stability of the results. The numerical approximation is defined by

$$\begin{aligned} &\frac{p(x_i,\nu_{j+1}) - p(x_i,\nu_j)}{\tau} = \\ &\frac{\sigma^2}{4}x_i^2\,\frac{p(x_{i+1},\nu_{j+1}) - 2p(x_i,\nu_{j+1}) + p(x_{i-1},\nu_{j+1})}{h^2} + \\ &\frac{\sigma^2}{4}x_i^2\,\frac{p(x_{i+1},\nu_j) - 2p(x_i,\nu_j) + p(x_{i-1},\nu_j)}{h^2} - \\ &f(x_i)\frac{p(x_{i+1},\nu_{j+1}) - p(x_{i-1},\nu_{j+1})}{2h} \end{aligned} \quad (10)$$

for the multiplicative noise, and

$$\begin{aligned} &\frac{p(x_i,\nu_{j+1}) - p(x_i,\nu_j)}{\tau} = \\ &\frac{\sigma^2}{4}\,\frac{p(x_{i+1},\nu_{j+1}) - 2p(x_i,\nu_{j+1}) + p(x_{i-1},\nu_{j+1})}{h^2} + \\ &\frac{\sigma^2}{4}\,\frac{p(x_{i+1},\nu_j) - 2p(x_i,\nu_j) + p(x_{i-1},\nu_j)}{h^2} - \\ &f(x_i)\frac{p(x_{i+1},\nu_{j+1}) - p(x_{i-1},\nu_{j+1})}{2h} \end{aligned}$$

for the additive noise

$i = 1,2,\ldots,N-1,\ h = \dfrac{x_{\sup}}{N}$,

$j = 0,1,\ldots,M-1,\ \tau = \dfrac{\nu_{\sup}}{M}$.

The initial and boundary conditions are as follows

$$\begin{aligned} &p(0,\nu) = 0,\ \nu \in [0;\nu_{\sup}], \\ &p(x_{\sup},\nu) = 0,\ \nu \in [0;\nu_{\sup}], \\ &p(0,0) = \delta(0), \end{aligned} \quad (11)$$

$\delta(\cdot)$ is a Dirac function, M, N are the numbers of steps.

As was above-mentioned the objective of our study is to develop the diameter distribution at any point in the age - height - basal area - stand density - site index space. So, the probability density function of tree diameter can be described by means of the mixture

$$p(x,\nu_1,\nu_2,\ldots,\nu_k) = \sum_{i=1}^{k} q_i p_i(x,\nu_i) \quad (12)$$

$$\sum_{j=1}^{k} q_j = 1,$$

where k is the number of stand-level characteristics (age, height, basal area, stand density, site index), q_i $i = 1,2,\ldots,k$ denote the weight of ith stand-level characteristic's, $p_i(x,\nu_i)$ $i = 1,2,\ldots,k$ is the ν_i-dependent transition probability density function of diameter. In the sequel we address to two stand level characteristics age and height. The stand-level characteristics age and height can be subdivided into the some age and height classes. So, the set of overlapping components of diameter distribution can be compounded by the joint distribution of the following form

$$p(x) = q_1 \sum_{j=1}^{m_1} \lambda_j p_1(x, t_j) + q_2 \sum_{j=1}^{m_2} w_j p_2(x, h_j), \qquad (13)$$

$$\sum_{j=1}^{2} q_j = 1, \ \sum_{j=1}^{m_1} \lambda_j = 1, \ \sum_{j=1}^{m_2} w_j = 1,$$

where m_1 is the number of groups according to the age classes, m_2 is the number of groups according to the height classes, λ_j is the part of the stand with j-age class trees, w_j is the part of the stand with j-height class trees.

For the evaluation of our derived mixtures (12), (13) three goodness of fit tests were computed as follows:

1. The Shapiro and Francia statistic to quantify the straightness of a normal probability plot is defined by the squared sample correlation between the pairs of points (r_i, q_i), $i = 1,2,...,n$ [52]. The pseudo-residuals, r_i, corresponding to the observation (d_i, t_i, h_i) are defined in the following form

$$r_i = \Phi^{-1}\left(\int_0^{d_i} p(x, t_i, h_i) dx \right), \ i = 1,2,...,n,$$

where: Φ denotes the distribution function of the standard normal distribution, (d_j, t_j, h_j) is the j-th observation of diameter, age and height. The ordered pseudo-residuals, $r_{(i)}$ $i = 1,2,...,n$, are plotted against the plotting positions

$$q_i = \Phi^{-1}\left(\frac{i}{n+1} \right), \ i = 1,2,...,n.$$

The key assumptions are that the observations X_1, X_2, …, X_n are independent and have the distribution function $F_i(x)$ (note, that X_i are not assumed to be identically distributed).

2. The Kolmogorov-Smirnov supremum class statistic (*KS*) is defined by Kolmogorov [25]

$$KS = \sqrt{n} \max\left\{ \sup_{1 \le j \le n}\left\{ \frac{j}{n} - u_{(j)} \right\} \sup_{1 \le j \le n}\left\{ u_{(j)} - \frac{j-1}{n} \right\} \right\},$$

where:

$$u_j = \int_0^{d_j} p(x, t_j, h_j) dx, \ j = 1,2,...,n,$$

$u_{(j)}$ are ordered values.

3. Cramer - von Mises quadratic class statistic (*CM*) is defined by Cramer [10]

$$CM = \frac{n}{3} + \frac{1}{n} \sum_{j=1}^{n} (1 - 2j) u_{(j)} + \sum_{j=1}^{n} (u_{(j)})^2.$$

The normal probability plot of pseudo-residuals enables us to evaluate visually the fit of diameter distribution to the observations. Small values of the test statistics *KS*, *CM* and large p-values indicate a better fit.

3 Results

We have demonstrated new theoretical results on the diameter distribution problem in the presence of both additive and multiplicative noises. We now illustrate these results by characterizing the underlying diameter dynamics as a stochastic process with the multiplicative noise. For model estimation were used observations of 1581 pines. These observations are represented in Figure 1. The estimations of the parameters r, K, β, σ for the all used stochastic logistic growth laws were calculated using the maximum likelihood procedure [48]. These results are represented in Table 3. A comparative analysis for the performance of the stochastic logistic growth laws (2) for the diameter growth modeling was carried out on the basis of the coefficient of determination. A confidence interval for the coefficient of determination, R^2, was calculated using a Monte Carlo method. In order to simulate the stochastic logistic growth laws (2) we used the Ito stochastic calculus and the second-order Milshtein difference scheme [36] defined in the following form

$$X_{n+1} = X_n + f(X_n)\Delta + \sigma(X_n)\Delta W_n + \frac{1}{2}\sigma(X_n)\frac{\partial}{\partial X_n}\sigma(X_n)\left((\Delta W_n)^2 - \Delta\right),$$

where: Δ is the time step size, $W_0 = 0$, $\Delta W_j = W_{j-1} + Y_j$, $j = 1,2,...,L$, Y_j is an independent random variable of the form $\sqrt{\Delta} N(0,1)$, L is the number of steps.

This procedure was repeated 100 times to obtain a set of replicates of averaged numerical approximations (average of 1000 simulations of equation (2) with an integration step-size $\Delta = 1$ year or $\Delta = 0.2$ meters). Finally, the confidence interval is defined as percentiles of the simulated distribution function $\hat{F}(r)$ of the coefficient of determination. A Monte Carlo estimation of the distribution function $\hat{F}(r)$ of R^2 is defined by

$$\hat{F}(r) = \#(R_i^2 \le r)/100,$$

where: # means “the number of”. The $100\gamma\%$ confidence interval for R^2 is given by

$$\left[R_L^2;\quad R_U^2\right]=\left[\hat{F}^{-1}\left(\frac{1-\gamma}{2}\right);\ \hat{F}^{-1}\left(\frac{1+\gamma}{2}\right)\right].$$

The results of confidence intervals of the coefficient of determination are presented in Table 3.

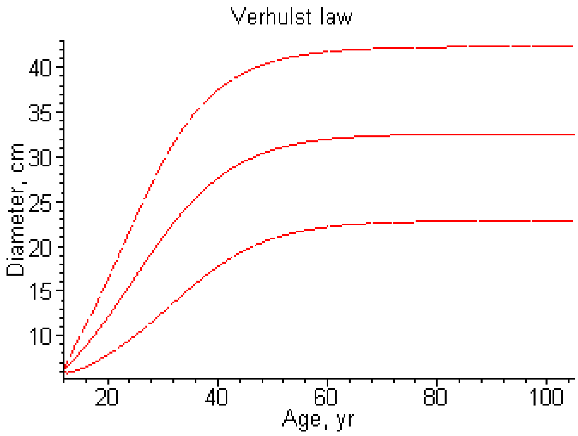
On the other side, the mean trajectory of diameter growth and its standard deviation can be also calculated from the set of equations (4)-(8). The numerical solutions of the set of equations (4)-(8) are shown in Figure 2. The fit of the mean ($m_\nu = E(X(\nu))$) calculated by the set of equations (4)-(8) to data is presented in Table 4. As was expected, both methods the second-order Milshtein difference scheme and the set of equations (4)-(8) for the mean diameter growth modeling have about the same explanatory power, because the fitting measures are very close (see, Tables 3, 4). In both cases the mean $m_\nu = E(X(\nu))$ and standard deviation $\sqrt{s_\nu} = \sqrt{V(X(\nu))}$ of diameter evolve to steady-state mean and standard deviation (see, Tables 3, 4). The steady-state mean and standard deviation of diameter size were calculated using the steady-state probability density functions (see, Table 2). The values of steady-state mean and standard deviation are presented in Table 4. The steady-state densities (see, Table 2) of the above-mentioned stochastic logistic growth laws are also represented in Figure 3 (non-continuous curve). All five stochastic logistic growth laws have similar types of steady-state probability density functions following bell-shaped and left-truncated (Figure 3). The numerical approximations of probability density functions of diameter size were calculated by equations (10)-(11). For all five stochastic logistic diameter growth laws the numerical approximations of equation (3) are presented in Figure 3.

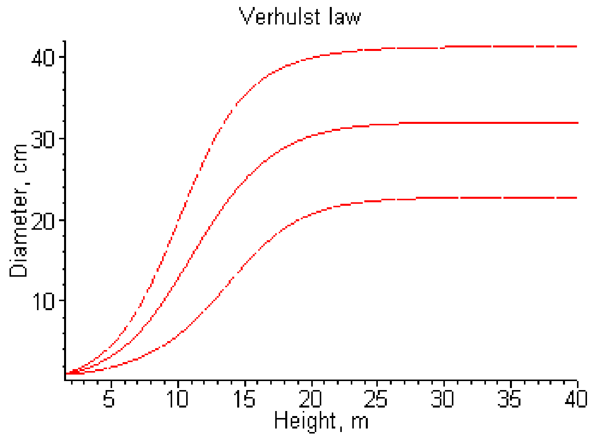
Table 3. Estimations of parameters.

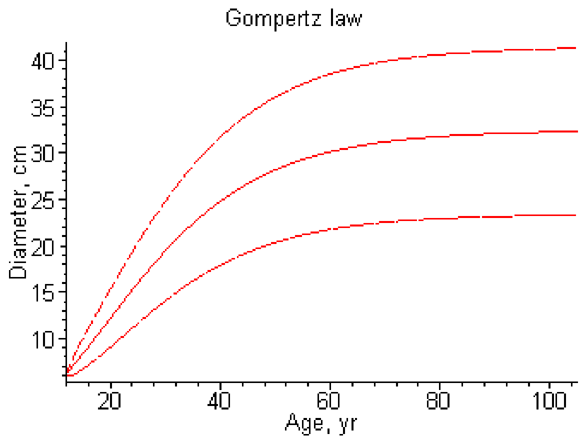
Law	Case	Parameters				Coefficient of determination R^2 $\gamma = 0.95$		
		r	K	β	σ	R^2_{lower}	R^2_{mean}	R^2_{upper}
Verhulst	Age	0.1133	32.8650		0.1354	0.6604	0.6887	0.7078
	Height	0.3637	33.8910		0.2230	0.7312	0.7613	0.7826
Gompertz	Age	0.0647	33.6718		0.1350	0.6711	0.7045	0.7281
	Height	0.2087	30.9075		0.1790	0.7359	0.7602	0.7942
Mitscherlich	Age	0.9313	42.4589		0.1350	0.6643	0.7272	0.7614
	Height	1.9178	70.3583		0.1468	0.7523	0.8046	0.8435
Bertalanffy	Age	0.3866	40.2436		0.1339	0.7057	0.7482	0.7804
	Height	1.3937	32.2951		0.1580	0.7575	0.7834	0.8043
Richards	Age	0.1033	32.9119	1.1966	0.1371	0.6720	0.6961	0.7313
	Height	1.0054	31.0288	0.2509	0.1908	0.7186	0.7499	0.7718

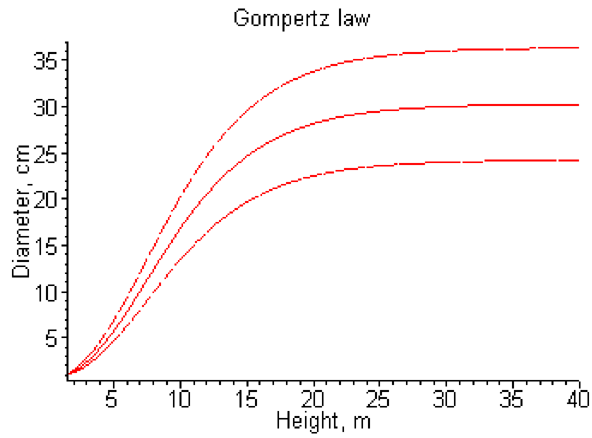
Table 4. Steady-state mean, standard deviation and determination coefficient for the fit of the mean $m_\nu = E(X(\nu))$.

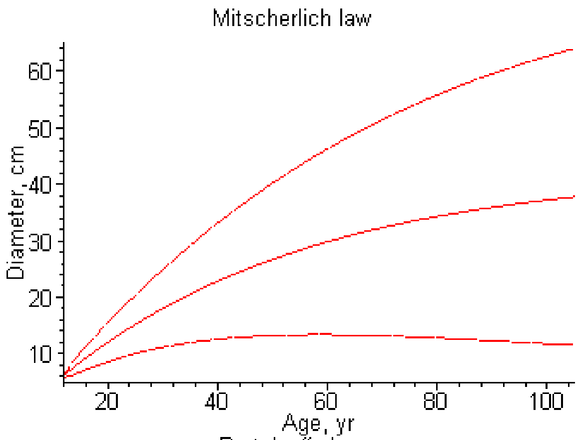
Law	Case	Mean	Standard deviation	Coefficient of determination
Verhulst	Age	30.2595	8.9520	0.6451
	Height	31.5740	8.5531	0.7368
Gompertz	Age	33.6091	12.7155	0.7136
	Height	30.9072	8.6450	0.7574
Mitscherlich	Age	35.0342	18.4557	0.7517
	Height	43.7879	20.0409	0.8170
Bertalanffy	Age	35.6389	16.5739	0,7313
	Height	31.3891	9.3925	0.7874
Richards	Age	30.1447	8.7247	0.6302
	Height	29.6503	8.1202	0.7406

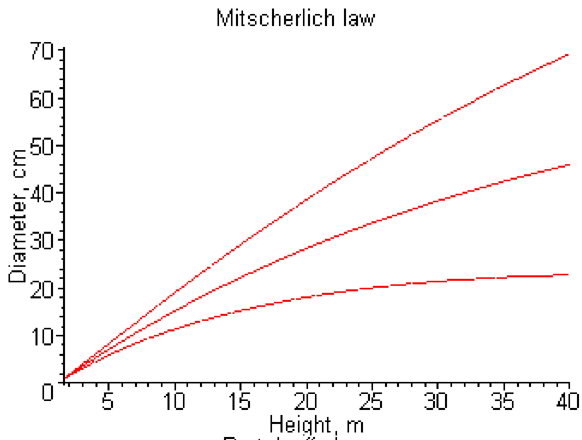

Verhulst law
Diameter, cm
Age, yr

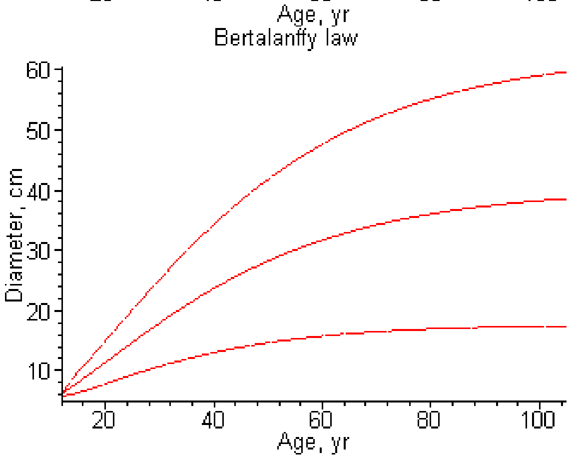

Verhulst law
Diameter, cm
Height, m

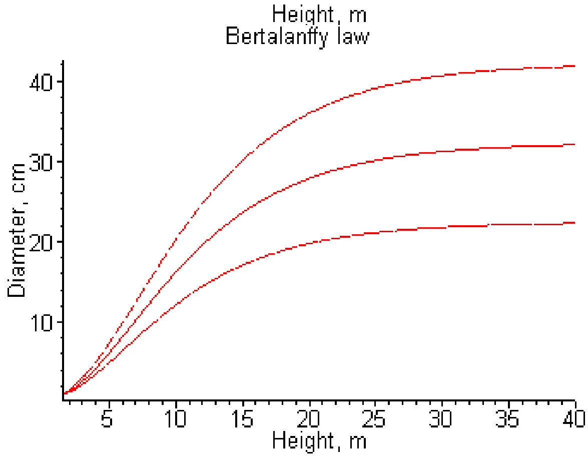

Gompertz law
Diameter, cm
Age, yr


Gompertz law
Diameter, cm
Height, m


Mitscherlich law
Diameter, cm
Age, yr


Mitscherlich law
Diameter, cm
Height, m


Bertalanffy law
Diameter, cm
Age, yr


Bertalanffy law
Diameter, cm
Height, m

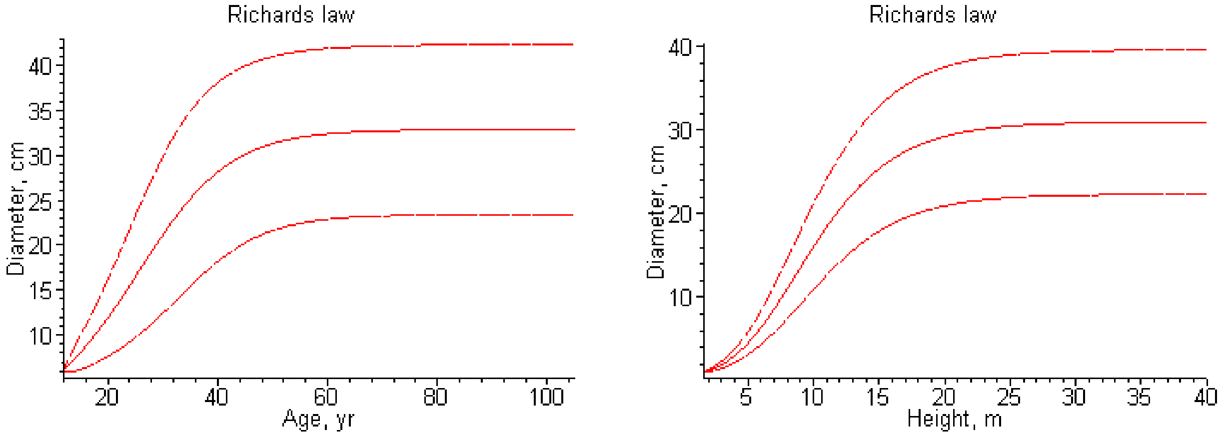
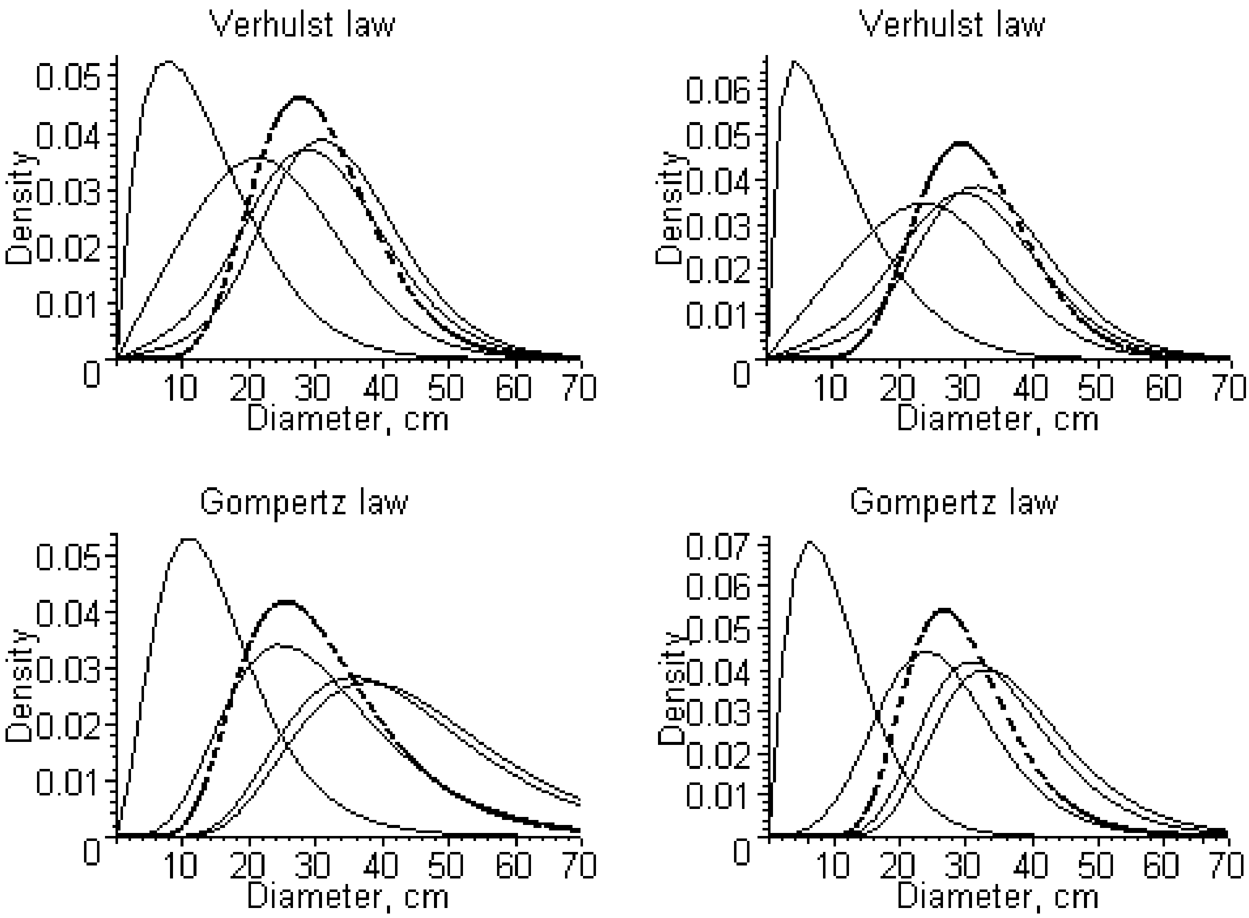


Fig. 2. Numerical solutions of the set of equations (4)-(8): mean (continuous curve), mean $\pm$ diameter's standard deviation (non-continuous curve).



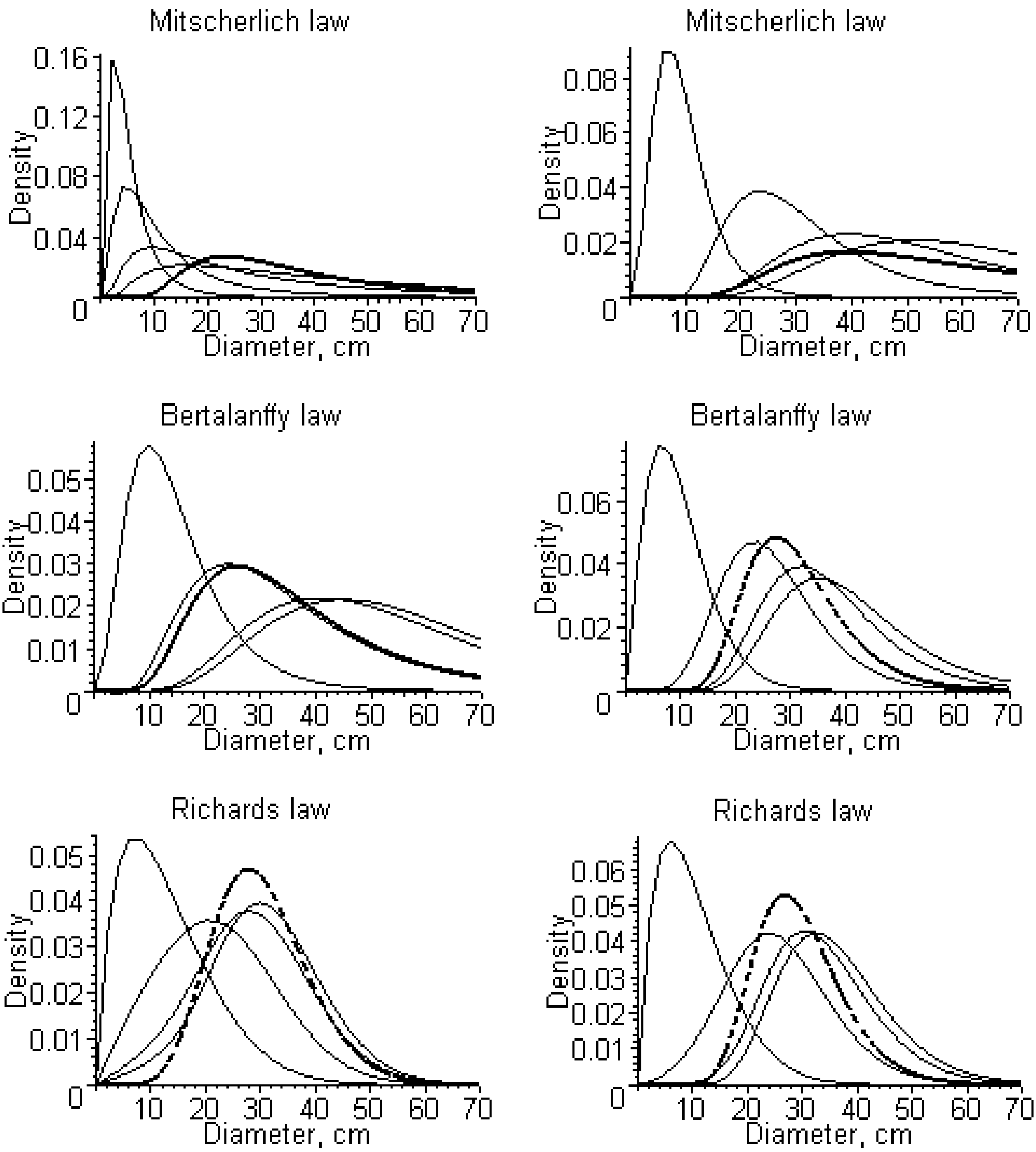


Fig. 3. **N**umerical approximations of diameter's probability density functions (on the left at the age 20, 40, 60, 80 yr; on the right at the height of 5, 15, 25, 35 m) and steady-state solution (non-continuous curve).

These approximations have different shapes at various stages of stand-level development (age, height). For the modeling of the age-dependent probability density function $p_1(x,t)$ of diameter distribution, the von Bertalanffy stochastic growth law was found to be the most suitable predictor (Table 3). In the height-dependent probability density function $p_2(x,h)$ of diameter distribution, the Mitscherlich growth law was more suited predictor. These two probability density functions are presented in Figure 4.

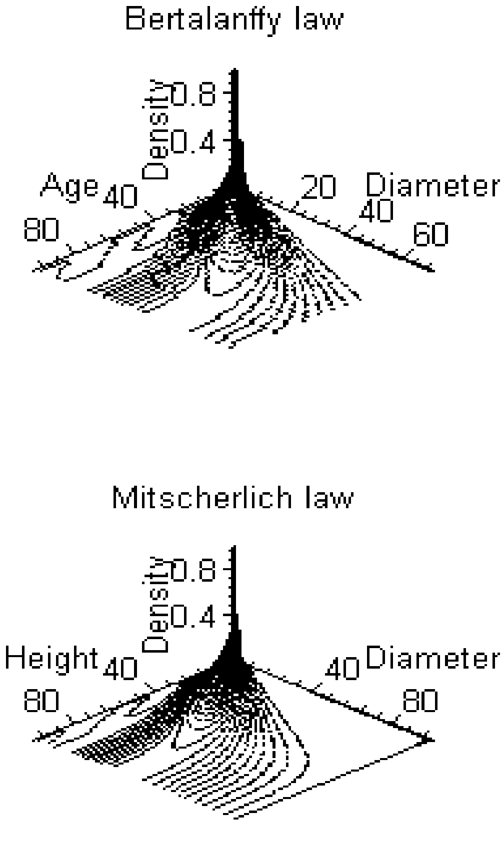


Fig. 4. **N**umerical approximations of diameter probability density function.

To further data analysis 11 equal (10 years) age intervals and 9 (4 meters) height intervals were set. Consequently, we have $m_1 = 11, m_2 = 9$. According to 1581 tree observation data the weight coefficients $\lambda_i, i = 1,2,...,11$, $w_j, j = 1,2,...,9$ were calculated and presented in Table 5. In this table fixed mean values of age and height classes, which are used for calculation of probability density functions, were presented as well. In Figure 5 density mixtures (13) depending on age and height are presented as well as joint mixture depending on both age and height (the age-dependent von Bertalanffy density and the height-dependent Mitscherlich density). It is worth mentioning that the difference between these diameter probability density function mixtures is very slight.

The normal probability plots of the corresponding pseudo-residuals, using the estimates of parameter given in Table 3 and the density mixture (12), are shown in Figure 6. The plots in Figure 6 show that the density mixture (12) fits the fitting data set (see, Fig. 1) comparatively well for the all used diameter growth laws. The simulated 1% confidence intervals (see, Fig. 6) and calculated the Shapiro-Francia statistics 0.9938 (for theVerhulst law), 0.9946 (for the Gompertz law), 0.9929 (for the Mitscherlich law), 0.9855 (for the von Berlalanffy law), 0.9901 (for the Richards law) lead us to a conclusion that the data are compatible with the density mixture (12). It should be remembered that the fitting data set was sufficiently large, and may create also other problems, namely, a statistical test may failed due too large sample.

For evaluating density mixture (12) we used a validation data set, which consists of 61 measurements. The normal probability plots for the validation data set, with the parameter estimates given in Table 3, are shown in Figure 7. The summaries of the Shapiro-Francia, Kolmogorov-Smirnov and Cramer - von Mises goodness of fit statistics are presented in Table 6. Hence, all estimated density mixtures (12) of diameter distribution models (2) didn't differ from the empirical distribution by the Shapiro-Francia statistic at 5% confidence level. The normal probability plots in Figure 7 show that the all estimated density mixtures (12) of diameter distribution fit the validation data set very well at 5% confidence level. Next, the estimated density mixtures (12) of the Richards and Verhulst type's were likely exemplars by the Kolmogorov-Smirnov test at 5% confidence level (Table 6). Ultimately, the Richards and Verhulst estimated density mixtures (12) fitted the validation data set by the Cramer - von Mises test at 1% confidence level.

Table 5. Weights and mean values of the density mixture (13).

Weight and mean value	Class									
	1	2	3	4	5	6	7	8	9	10
λ_i	0..4358	0.0556	0.0859	0.0872	0.0960	0.1106	0.0720	0.0404	0.0114	0.0051
t_i	16	26	34	44	54	64	74	84	94	102
w_i	0.0764	0.3619	0.0499	0.0948	0.1074	0.1276	0.1295	0.0474	0.0051	
h_i	3	5	10	14	18	22	26	29	33	

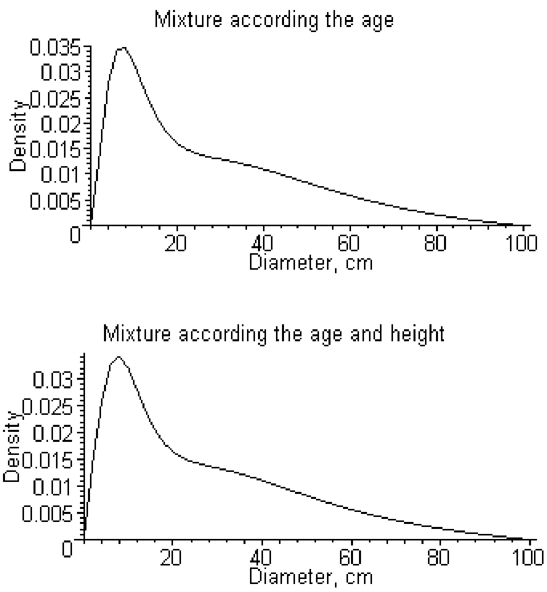
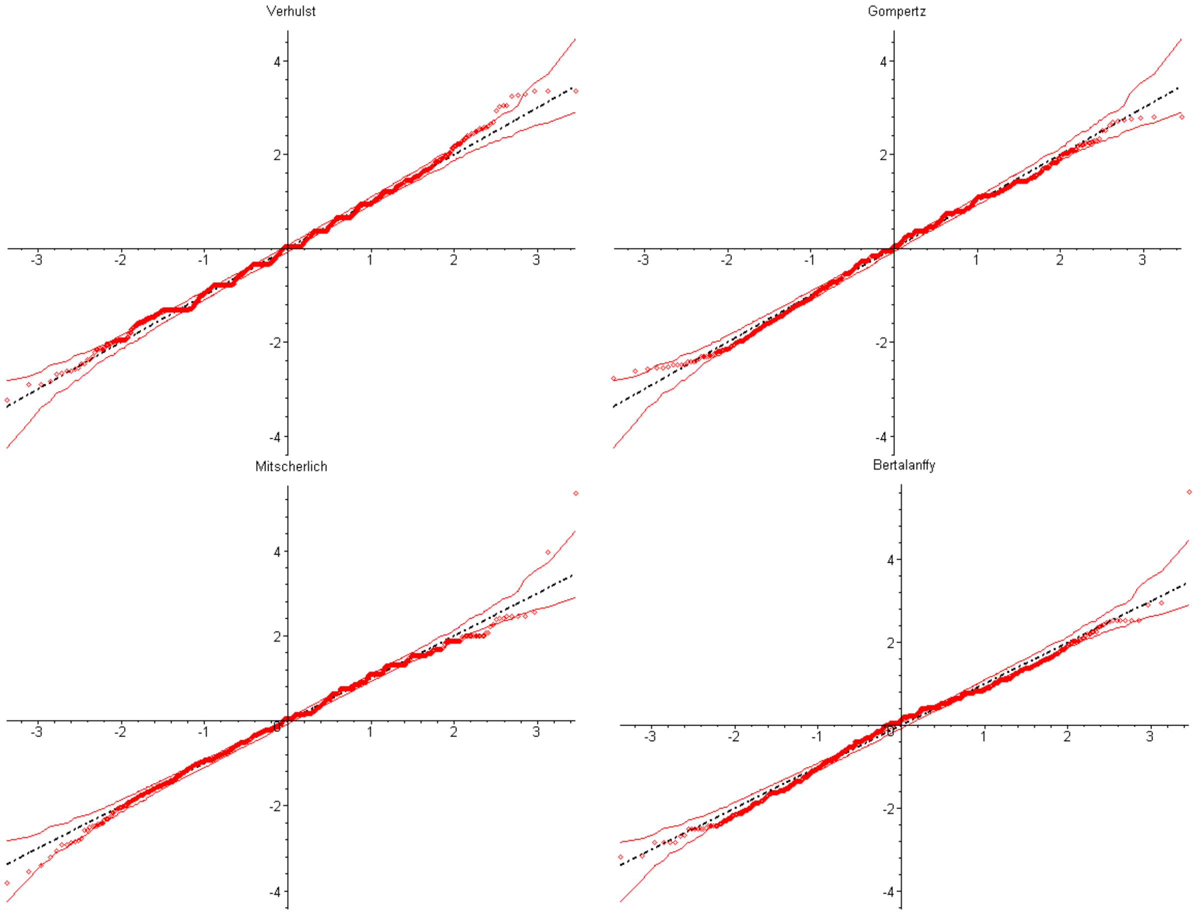


Fig. 5. Mixture (12) of the probability density functions.

Obviously, it is also possible to improve the fit of such age and height dependent estimated mixture (12) of probability density functions by using a mixture of three or more stand-level characteristics (basal-area, stand density, site index) instead of just two.



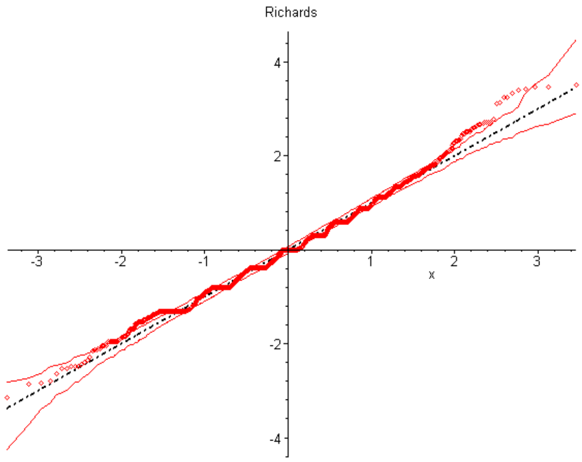
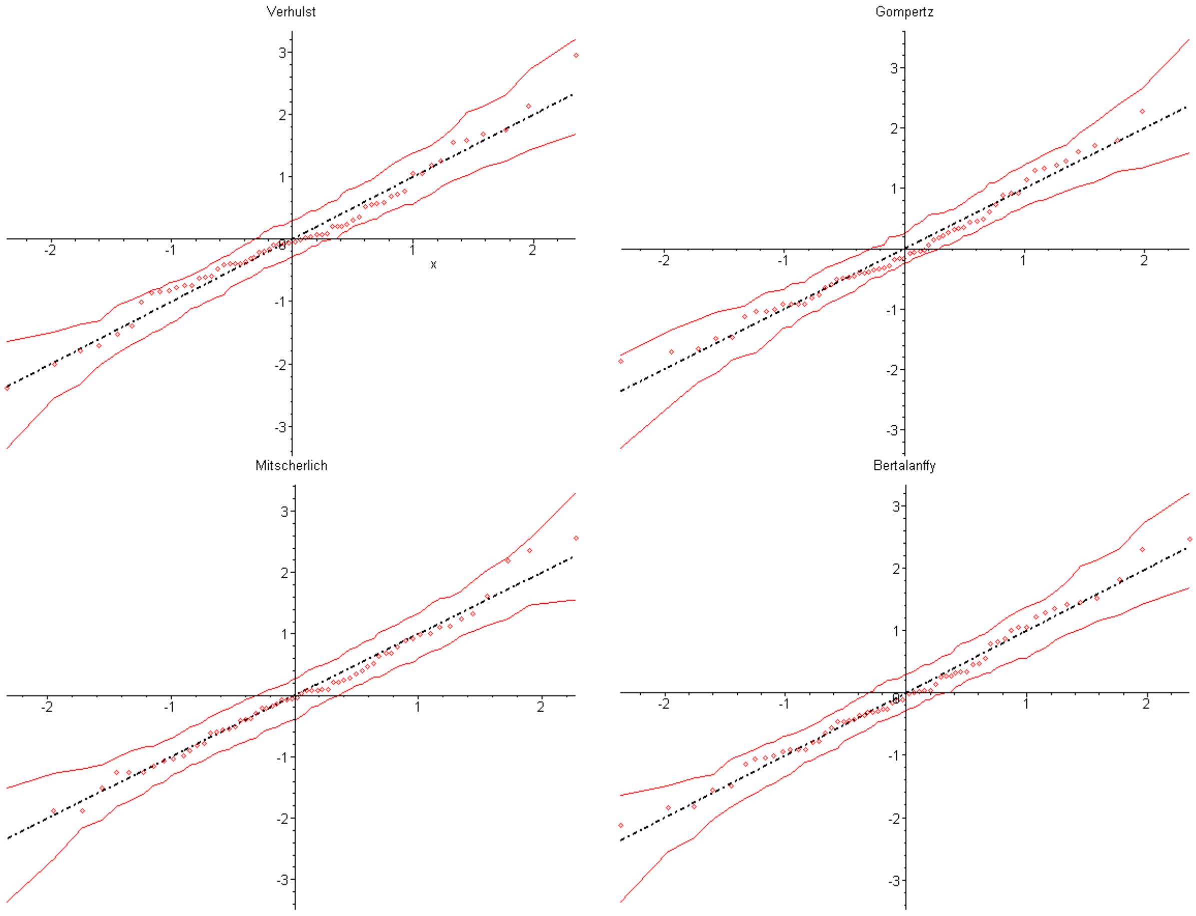


Fig. 6. Normal probability plots of pseudo-residuals for all used diameter growth laws and fitting data set.



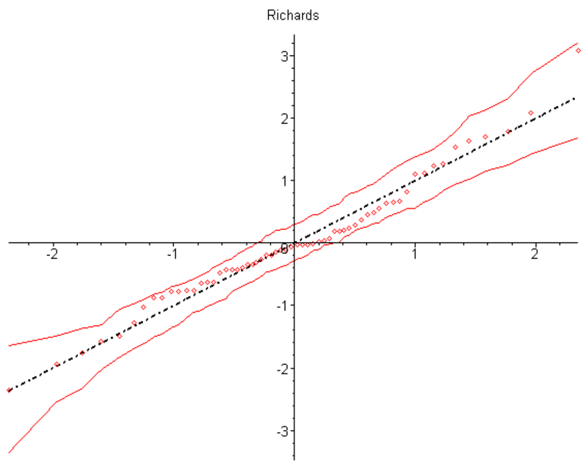


Fig. 7. Normal probability plots of pseudo-residuals for all used diameter growth laws and validation data set.

Table 6. Goodness of fit statistics calculated to the validation data set.

Law	Statistics		
	Shapiro-Francia (p-value)	Kolmogorov-Smirnov (p-value)	Cramer-von Mises (p-value)
Verhulst	0.9760 (0.2351)	0.1854 (0.2451)	0.6842 (0.0140)
Gompertz	0.9801 (0.339)	0.3861 (0.0002)	3.0375 (<0.0001)
Mitscherlich	0.9849 (0.550)	0.3343 (0.0022)	2.2084 (<0.0001)
Bertalanffy	0.9892 (0.788)	0.4430 (<0.0001)	4.1419 (<0.0001)
Richards	0.9730 (0.157)	0.1749 (0.3083)	0.5510 (0.0297)

4 Discussion

Although our interest in this article was to introduce methodology rather than analysis models, we make same remarks on the distinction between the stochastic logistic growth model (2) and its deterministic counterpart, defined by (1).

As we can see from equations (4)-(8), the mean diameter size is extremely sensitive with respect to the form and size of the coefficient of volatility σ in the case of the Verhulst, Gompertz, von Bertalanffy and Richards ($\beta > 0$) laws. It is clear that the deterministic approximations of these four growth laws overestimate the true mean diameter size in the presence of stochastic perturbations.

For the Mitscherlich (linear) growth law, the volatility σ does not affect the mean diameter size. The comparative dynamic analysis of equations (4)-(8) shows that the mean of trajectory of diameter decreases when the intensity of noise σ increases (Verhulst, Gompertz, von Bertalanffy and Richards laws), but the variance of diameter increases for the all used laws.

An analysis of Table 3 contrasts the performance of the height dependent stochastic logistic diameter growth laws against the age dependent stochastic logistic diameter growth laws (see, Table 3). The results convincingly support of the use of the diameter and height growth equations. This is consistent with the results obtained in [53], [58], [67]. From the results given in Table 3 it is evident that von Bertalanffy's law is best applied when simulating the stand diameter reliance on age, and Mitscherlich's law is most suitable for simulating the stand diameter reliance on height. It should be noted that the coefficient of determination of the analyzed models (2) doesn't fix essential differences in suitability.

Diameter-height-age growth equations provide an adequate description of stands growth. Note that the mean of diameter's trajectory monotonically evolves toward the steady-state value for the all used stochastic logistic growth laws (see, Table 4 and Figure 2). But the path by which the variance of diameter's trajectory evolves to the steady-state value not always increasing for the Verhulst, Gompertz, von Bertalanffy and Richards laws. Depending on the initial conditions, the variance of diameter's trajectory could quickly increase above its steady-state value, reach its maximum at a value of predictor variable when the rate of growth is near its maximum, and then decrease to the steady–state.

Next we discuss the numerical approximations of the diameter probability density functions $p_1(x,t)$ (at the age 20, 40, 60, 80 years), $p_2(x,h)$ (at the height of 5, 15, 25, 35 metres) and the steady-state probability density functions $p_1(x)$, $p_2(x)$. The probability density functions $p_1(x,t)$, $p_2(x,h)$ have different shapes at various stages of stand-level development (age, height). All five numerical approximations of diameter probability density functions $p_1(x,t)$ (at the age 20, 40, 60, 80 years), $p_2(x,h)$ (at the height of 5, 15, 25, 35 metres) and the steady-state probability density functions $p_1(x)$, $p_2(x)$ have similar type following bell-shaped and left-truncated (Figure 3). The von Bertalanffy stochastic growth law was found to be the most suitable predictor for the modeling of age-dependent probability density function $p_1(x,t)$ (Table 3). The Mitscherlich growth law was more suited in the modeling of the height-dependent probability density function $p_2(x,h)$ of diameter distribution. As we can see from Figure 3 the multi-modality of diameter distribution could be revealed using the mixtures (12), (13) of probability density functions.

5 Conclusions

In this paper an original probability density functions of tree diameter distribution are presented. The stochastic population model that we have considered is one of the several possible stochastic versions of the corresponding deterministic logistic population growth in a random fluctuating environment characterized by white noise. Obviously no model can accurately describe every biological phenomenon that foresters encounter in their practice and the same is true for our derived probability density functions of diameter size. It is an important problem to find correct criteria to decide is a good given stochastic model? Many probability density functions have been adjusted to deal with diameter distribution and new ones are continuously being proposed.

The purpose of this paper was to introduce a new method for the fitting of diameter distribution and show how this novelty can be implemented. For a realistic representation of diameter growth, we used linear and nonlinear drift functions and noise in the additive and multiplicative forms. The results obtained here have shown that it is possible to relate nonlinear stochastic diameter growth law and diameter distribution law. We investigated this relationship in the case of diameter growth dynamics expressed by the Verhulst, Gompertz, Mitscherlich, von Bertalanffy and Richards growth laws. The trajectory of diameter is under the influence of the shape of trend and the stochastic variation in the observed data set, as the parameters of diameter growth law were estimated by the method of maximum likelihood procedure. Two types of probability density functions of diameter size were obtained: steady-state and dependent on stand level characteristic. The steady-state probability density functions were presented in the exact form (Table 2) and age-dependent or height-dependent probability density functions were simulated numerically. The results of computer simulation showed that the age-dependent, height-dependent probability density functions and their mixture can be realized easily.

Our derived entire stand-level dependent diameter distribution is naturally more precise and needful then diameter distribution based on mean value of stand-level characteristic. Particularly, the stand-level dependent diameter distribution comes in useful modeling the problems of forest management. As an instance of the competence of the diameter distribution at various stages of stand development is its utility in: (1) designing thinning operations, (2) estimating the number of trees in each discrete diameter class at any point during stand development, (3) evaluating the economical impact of alternative thinning regimes. The advantage of these probability density functions obtained theoretically from the stochastic logistic diameter growth laws can be validated comparing with commonly used diameter distributions such as Weibull, negative exponential an so on.

While the presented probability density functions of diameter distribution capture the essential random behavior of diameter growth and give results that correspond to the data, they suffer from the fact that they are computationally expensive.

A theoretical prerequisite of our presented approach was the stochastic logistic diameter growth law. Thus, the proposed method could be continued modifying the drift and diffusion functions of the stochastic logistic diameter growth process (2). In order to propose more precise stochastic growth laws the stochastic delay differential equations could be used too [14], [50].

Our proposed method provides to derive the probability density function of diameter distribution which depends on a select part of the stand-level characteristics.

Finally, it is interesting to consider an alternative information theoretic approach of modeling and assessing of dynamics of stands diameter. The information theoretical measures play a crucial theoretical role in physics of macroscopic equilibrium systems. The Shannon's entropy and Fisher's information represent promising tools to illustrate the behavior of multidimensional systems in biology, ecology.

Lastly, it is our hope that this modeling effort will be a spring-board to practical studies that further improve the prediction accuracy of the model.

References

[1] L.H.R. Alvarez, E. Koskela, Wicksellian Theory of Forest Rotation under Interest Rate Variability, *J. Econ. Dynam. Control*, 29, 2005, pp. 529-545.

[2] M.S. Bartlett, J.C. Gower, P.H. Leslie, A Comparison of Theoretical and Empirical Results for Some Stochastic Population Models, *Biometrika*, 47, 1960, pp. 1-11.

[3] B.P. Bullock, E.L. Boone, Deriving Tree Distributions Using Bayesian Model Averaging, *For. Ecol. Manage.*, 242, 2007, pp. 127-132.

[4] B.J. Cairns, J.V. Ross, T. Taimre, A Comparison of Models for Predicting Population Persistence, *Ecol. Model.*, 201, 2007, pp. 19-26.

[5] R. Calama, G. Montero, Interregional Nonlinear Height-diameter Model with Random Coefficients for Scots Pine in Spain, *Can. J. For. Res.*, 34, 2004, pp. 150-163.

[6] D.A. Charlebois, A.S. Ribeiro, A. Lehmussola, J. Lloid-Price, O. Yli-Harja, S.A. Kauffman, Effects of Microarray Noise on Inference Efficiency of a Stochastic Model of Gene Networks, *WSEAS Transactions on Biology and Biomedicine*, Vol. 4, Issue 2, February 2007, pp. 15-21.

[7] W. Chen, Tree Size Distribution Functions of Four Boreal Forest Types for Biomass Mapping, *For. Sci.*, 50, 2004, pp. 436-449.

[8] Z. Chladna, Determination of Optimal Rotation Period under Stochastic Wood and Carbon Prices, *Forest Policy and Economics*, 9, 2007, pp. 1031-1045.

[9] H.R. Clarke, W.J. Reed, The Tree Cutting Problem in a Stochastic Environment: the Case of Age Dependent Growth, *J. Econ. Dynam. Control*, 13, 1989, pp. 569-595.

[10] H. Cramer, On the Composition of Elementary Errors. Second Paper: Statistical Applications, *Skand. Actuartidskr.*, 11, 1928, pp. 141-180.

[11] U. Dieguez-Aranda, F.C. Dorado, J.G. Alvarez Gonzalez, A.R. Alboreca, Dynamic Growth Model for Scots Pine (*Pinus sylvestris L.*) Plantations in Galicia (North-western Spain), *Ecol. Model.*, 191, 2005, pp. 225-242.

[12] M. Falconi, On the Dynamics of Age-dependent Predation Models, *WSEAS Transactions on Biology and Biomedicine*, Vol. 2, Issue 1, January 2005, pp. 32-35.

[13] J.C. Fox, P.K. Ades, H. Bi, Stochastic Structure and Individual-tree Growth Models, *For. Ecol. Manage.*, 154, 2001, pp. 261-276.

[14] T.D. Frank, Delay Fokker-Plank Equations, Novikov's Theorem, and Boltzmann Distributions as Small Delay Approximations, *Physical Review E*, 72, 2005, pp. 011112.

[15] V. Galvanauskas, R. Simutis, Software Tool for Efficient Hybrid Model-based Design of Biochemical Processes, *WSEAS Transactions on Biology and Biomedicine*, Vol. 4, Issue 9, September 2007, pp. 136-144.

[16] O. Garcia, A stochastic Differential Equation Model for the Height Growth of Forest Stands, *Biometrics*, 39, 1983, pp. 1059-1072.

[17] O. Garcia, The State–space Approach in Growth Modelling, *Can. J. For. Res.*, 24, 1994, pp. 1894 – 1903.

[18] O. Garcia, F. Ruiz, A Growth Model for Eucalypt in Galicia, Spain, *For. Ecol. Manage.*, 173, 2003, pp. 49-62.

[19] O. Garcia, Comparing and Combining Stem Analysis and Permanent Sample Plot Data in Site Index Models, *For. Sci.*, 51, 2005, pp. 277-283.

[20] I.I. Gihman, A.V. Skorohod, *Stochastic Differential Equations*, Springer-Verlag, New York.

[21] B. Gompertz, On the Nature of the Function Expressive of the Law of Human Mortality, and on a New Mode of Determining the Value of Life Contingencies, *Phil. Trans. Roy. Soc.*, 115, 1825, pp. 513-585.

[22} J. Guardiola, A.Vecchio, A Distributed Delay Model of Viral Dynamics, *WSEAS Transactions on Biology and Biomedicine*, Vol. 2, Issue 1, January 2005, pp. 36-41.

[23] W.L. Hafley, H.T. Schreuder, Statistical Distributions for Fitting Diameter and Height Data in Even-aged Stands, *Can. J. For. Res.*, 4, 1977, pp. 481-487.

[24] J.P. Keener, Stochastic calcium oscillations, *Mathematical Medicine and Biology*, 23, 2006, pp. 1-25.

[25] A. Kolmogorov, Sulla determinazione empirica di una legge di distribuzione, *Giorn. Ist. Ital. Attuari*, 4, 1933, pp. 83-91.

[26] A Kuliešis, A. Kasperavičius, G. Kulbokas, M. Kvalkauskienė, *Lithuanian National Forest Inventory*, Naujasis lankas, Kaunas.

[27] W.A. Leak, Long-term Structural Change in Uneven-aged Northern Hardwoods, *For. Sci*., 42, 1996, pp. 160-165.

[28] Y.C. Lei, S.Y. Zhang, Features and Partial Derivatives of Bertalanffy-Richards Growth Model in Forestry, *Nonlinear Analysis: Modelling and Control*, 9, 2004, pp. 65-73.

[29] F. Li, L. Zhang, C.J. Davis, Modeling the Joint Distribution of Tree Diameters and Heights by Bivariate Generalized Beta Distribution, *For. Sci.*, 48, 2002, pp. 47-58.

[30] C. Liu, L. Zhang, C.J. Davis, D.S. Solomon, J.H. Grove, A Finite Mixture Model for Characterizing the Diameter Distribution of Mixed-species Forest Stands, *For. Sci.*, 48, 2002, pp. 653-661.

[31] M. Maltamo, A. Kangas, J. Uuttera, T. Torniainen, J. Saramaki, Comparison of Percentile Based Prediction Methods and the Weibull Distribution in Describing the Diameter Distribution of Heterogeneous Scots Pine Stands, *For. Ecol. Manage.*, 133, 2000, pp. 263-274.

[32] R. Mankin, E. Soika, A. Sauga, Double Temperature-enhanced Occupancy of Metastable States in Fluctuating Bistable Potentials, *4th WSEAS International Conference on Mathematical Biology and Ecology*, 2008, pp. 24-28.

[33] X. Mao, G. Marion, E. Renshaw, Environmental Brownian Noise Suppresses Explosions in Population Dynamics. *Stochastic Process. Appl.*, 97, 2002, pp. 95-110.

[34] J.H. Matis, T.R. Kiffe, E. Renshaw, J. Hassan, A simple Saddlepoint Approximation for the Equilibrium Distribution of the Stochastic Logistic Model of Population Growth, *Ecol. Model.*, 161, 2003, pp. 139-248.

[35] L. Mehtatalo, Localizing a Predicted Diameter Distribution Using Sample Information, *For. Sci.*, 51, 2005, pp. 292-303.

[36] G.N. Milshtein, A Method of Second-order Accuracy for the Integration of Stochastic Differential Equations, *Theor. Probab. Appl.*, 23, 1978, pp. 396-401.

[37] T. Neeff, J.R. dos Santos, A Growth Model for Secondary Forest in Central Amazonia, *For. Ecol. Manage.*, 216, 2005, pp. 270-282.

[38] T.C. Nelson, Diameter Distribution and Growth of Loblolly Pine, *For. Sci.*, 10, 1964, pp. 105-115.

[39] P.F. Newton, Y. Lei, S.Y. Zhang, Stand-level Diameter Distribution Yield Model for Black Spruce Plantations, *For. Ecol. Manage.*, 209, 2005, pp. 181-192.

[40] T. Nord-Larsen, Q.V. Cao, A Diameter Distribution Model for Even-aged Beech in Denmark, *For. Ecol. Manage.*, 231, 2006, pp. 218-225.

[41] M. Palahi, T. Pukkala, A. Trasobares, Modelling the Diameter Distribution of *Pinus sylvestris*, *Pinus nigra* and *Pinus halapensis* Forest Stands in Catalonia Using the Truncated Weibull Function, *Forestry*, 79, 2006, pp. 553-562.

[42] M. Palahi, T. Pukkala, E. Blasco, A. Trasobares, Comparison of Beta, Johnson's SB, Weibull and Truncated Weibull Functions for Modeling the Diameter Distribution of Forest Stands in Catalonia (North-east of Spain), *Eur. J. Forest Res.*, 126, 2007, pp. 563-571.

[43] M.J. Penttinen, Impact of Stochastic Price and Growth Processes on Optimal Rotation Age, *Eur. J. Forest Res.*, 125, 2006, pp. 335-343.

[44] A. Quarteroni, A. Valli, *Numerical Approximation of Partial Differential Equation*, Springer Series in Computational Mathematics, Springer, 1977.

[45] J. Quartieri, S. Steri, M. Guida, C. Guarnaccio, S. D'Ambrosio, A Biomathematical Study of a Controlled Birth and Death Process Describing

Malignancy, 4^{th} *WSEAS International Conference on Mathematical Biology and Ecology,* 2008, pp. 108-115.

[46] F.J. Richards, A Flexible Growth Function for Empirical Use, *J. Exp. Bot.*, 10, 1959, pp. 290-300.

[47] P. Rupšys, Logistic Growth with Markovian Jumping, *Vagos*, 65(18), pp. 133-140.

[48] P. Rupšys, On Parameter Eestimation for Stochastic Logistic Growth Laws through the Maximum Likelihood and L^1 Norm Procedures, *Lithuanian Statistics: Articles, Reports and Studies*, 42, 2005, pp. 49-60.

[49] P. Rupšys, E. Petrauskas, J. Mažeika, R. Deltuvas, The Gompertz Type Stochastic Growth Law and a Tree Diameter Distribution, *Baltic Forestry*, 13, 2007, pp. 197-206.

[50] P. Rupšys, Delayed Stochastic Logistic Growth Laws in Single-Species Population Growth Modeling, 4^{th} *WSEAS International Conference on Mathematical Biology and Ecology*, 2008, pp. 29-34.

[51] J. Segarra, J. Raventos, M.F. Acevedo, Growth of Tropical Savana Plants in Competition: A Shoot Population Model, *Ecol. Model.*, 189, 2005, pp. 270-288.

[52] S.S. Shapiro, R.S. Francia, An Approximate Analysis of Variance Test for Normality, *Journal of the American Statistical Association*, 67, 1972, pp. 215-216.

[53] M. Sharma, S.Y. Zhang, Height-diameter Models Using Stand Characteristics for *Pinus banksiana* and *Picea mariana*, *Scand. J. For. Res.*, 19, 2004, pp. 442-451.

[54] I. Shoji, Approximation of Continuous Time Stochastic Processes by a Local Linearization Method, *Math. Comput.*, 67, 1998, pp. 287-298.

[55] J. Siipelehto, Improving the Accurancy of Predicted Basal-area Diameter Distribution in Advanced Stands by Determining Stem Number, *Silva Fennica*, 33, 1999, pp. 281-301.

[56] J. Siipelehto, J. Siitonen, Degree of Previous Cutting in Explaining the Differences in Diameter Distributions Between Mature Managed and Natural Norway Spruce Forests, *Silva Fennica*, 38, 2004, pp. 425-435.

[57] H. Sorensen, Parametric Inference for Diffusion Processes Observed at Discrete Points in Time: A Survey, *International Statistical Review*, 72, 2004, pp. 337-354.

[58] H. Temesgen, K.V. Gadow, Generalized Height-diameter Models – an Application for Major Tree Species in Complex Stands of Interior British Columbia, *Eur. J. Forest Res.*, 123, 2004, pp. 45-51.

[59] A. Tsoularis, Reinforcement Learning in Predator-prey Interactions, *WSEAS Transactions on Biology and Biomedicine*, Vol. 2, Issue 2, April 2005, pp. 141-146.

[60] A. Tsoularis, Learning Strategies for a Predator Operating in Model-mimic-alternative Prey Environment, *WSEAS Transactions on Biology and Biomedicine*, Vol. 3, Issue 3, March 2006, pp. 244-248.

[61] A. Tsoularis, J. Wallace, Analysis of Logistic Growth Models, *Math. Biosci.*, 179, 2002, pp. 21-55.

[62] J. Uusitalo, V.P. Kivinen, Constructing Bivariate dbh/dead-branch Height Distribution of Pines for Use in Sawing Production Planning, *Scandinavian Journal of Forest Research*, 13, 1998, pp. 509-514.

[63] P.F. Verhulst, Notice sur la loi que la population suit dans son accroissement, *Curr. Math. Phys.*, 10, 1838, pp. 113-121.

[64] L. von Bertalanffy, A Quantitative Theory of Organic Growth, *Human Biol.*, 10, 1938, pp. 181-213.

[65] C. Westphal, N. Tremer, G. von Oheimb, J. Hanse, K. von Gadow, W. Hardtle, Is a Reverse J-shaped Diameter Distribution Universally Applicable in European Virgin Beech Forests? *For. Ecol. Manage.*, 223, 2006, pp. 75-83.

[66] Y. Willassen, The Stochastic Rotation Problem: A Generalization of Faustman's Formula to Stochastic Forest Growth, *J. Econ. Dynam. Control*, 22, 1998, pp. 573-596.

[67] A.R. Weiskittel, S.M. Garber, G.P. Johnson, D.A. Maguire, R.A. Monserud, Annualized Diameter and Height Growth Equations for Pacific Northwest Plantation-grown Douglas-fir, Western Hemlock, and Red Alder, *For. Ecol. Manage.*, 250, 2007, pp. 266–278.

[68] A. Yoshimoto, I. Shoji, Comparative Analysis of Stochastic Models for Financial Uncertainty in Forest Management, *For. Sci.*, 48, 2001, pp. 755-766.

[69] M. Zasada, C.J. Cieszewski, A Finite Mixture Distribution Approach for Characterizing Tree Diameter Distributions by Natural Social Class in Pure Even–aged Scots Pine Stands in Poland, *For. Ecol. Manage.*, 204, 2005, pp. 145–158.

[70] B. Zeide, Intrinsic Units in Growth Modeling, *Ecol. Model.*, 175, 2004, pp. 249-259.

[71] M. Zhou, J. Boungiorno, Nonlinearity and Noise Interaction in a Model of Forest Growth, *Ecol. Model.*, 180, 2004, pp. 291-304.

[72] M. Zhou, J. Boungiorno, Forest Landscape Management in a Stochastic Environment, with an Application to Mixed Loblolly Pine-hardwood Forests, *For. Ecol. Manage.*, 223, 2006, pp. 170-182.

[73] W. Zucchini, M. Schmidt, K. von Gadow, A Model for the Diameter-height Distribution in an Uneven-aged Beech Forest and a Method to Assess the Fit of Such Models, *Silva Fennica*, 35, 2001, pp. 169-182.