On the Intrinsic Limiting Zeros as the Sampling Period Tends to Zero

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Abstract : It is rigorously proved that the limiting discrete zeros located at $z=1$ as the sampling period tends to zero are limiting intrinsic zeros, namely, they do not appear if the continuous plant (i.e. a dynamic continuous-time system to be controlled by discretization techniques) is zero-free. To prove that known result, it is not assumed as usual in the literature, that the plant is approximated by an integrator of order equal to its relative degree as the sampling period tends to zero. It is also proved that limiting zeros at $z=1$ are present for any fractional zero-order hold (FROH) including the zero-order hold (ZOH) and the first-order hold (FOH) even when the continuous plant is of zero relative degree (i.e. biproper).

Key-Words:- Continuous-time systems, discrete systems, zero-order hold, first-order and fractional holds

1. Introduction

Limiting intrinsic discrete zeros have been proved to be located at $z=1$ as the sampling period $T \to 0$ [1-4] when the continuous plant is strictly proper. The result is obtained under any FROH if the continuous plant is strictly proper (i.e., there is no direct interconnection input-output gain). Such zeros are intrinsic, namely, they are due to the continuous plant zeros so that they do not appear if the continuous plant is zero-free. However, that result has been obtained by approximating the continuous plant of $n$ poles and $m$ zeros, then of relative degree $n_r = n - m$, by an integrator of $n_r$-th order as $T \to 0$, [1-2]. That approach has been justified heuristically by comparing the discretized equations to the continuous case by taking limits as $T \to 0$. However, there is a lack of rigor in that problem statement since the continuous plant generates a dynamics defined by a differential equation while the sampling and hold device generates a particular piecewise continuous forcing input for such a differential equation at the given sampling rate. This circumstance translates into a difference equation solution which defines the sampled values of the output sequence after integrating the solution trajectory from the differential dynamics. Note that limits in integrated functions cannot be moved in or out under the integral symbol for any integrated function as the arguments tends to some limit if one operates with mathematical rigor. The correct final result found through the above mentioned approach was that the intrinsic discrete zeros converge to unity as $T \to 0$ if the continuous plant is strictly proper. In this paper, we reformulate the result for intrinsic limiting zeros by using directly the difference equations which depend on the sampling period without using the integrator approach and, furthermore, such a result is extended to biproper plants.

2. Continuous and discrete plant

Consider the state-space realization $R = (A, b, c^T, d)$ of the linear single-input single-output continuous time-invariant plant

$$\begin{align*}
\dot{x}(t) &= Ax(t) + bu(t); \quad x(0) = x_0 \\
y(t) &= c^T x(t) + d u(t)
\end{align*}$$

(1)

where $x \in \mathbb{R}^n$ is the state vector and $u$ and $y$ are the scalar input and output. If the direct interconnection gain $d$ is zero (nonzero) then the continuous plant is strictly proper (biproper). If a FROH, see for instance [5], is used for the input with a sampling period $T$ then the plant input becomes:

$$u(t) = u_k + \beta \left[ \frac{u_k - u_{k-1}}{T} \right] (t - kT)$$

(2)

for all $t \in [kT, (k + 1)T)$, $k \geq 0$ integer where $v_k = v(kT)$ denotes the value of a sampled signal $v(t)$ at sampling instants. The inputs for ZOH and FOH follow from (2) with $\beta = 0$ and $\beta = 1$, respectively. Eqs. 1-2 yield at sampling instants:
\[ x_{k+1} = \Phi(T)x_k + \Gamma(\beta, T)u_k + \Gamma_1(\beta, T)u_{k-1} \]
\[ y_k = c^TX_k + du_k ; k \geq 0 \] (3)

where
\[ \Phi(T) = e^{AT} ; \Gamma(\beta, T) = \left(1 + \frac{\beta}{T}\right)\Gamma_0(T) \]
\[ \Gamma_1(\beta, T) = -\frac{\beta}{T}\Gamma_0(T) \]
\[ \Gamma_0(T) = \Phi(T)\left(\int_0^T e^{-\Lambda T}d\tau\right)b \] (4)

The substitution of \( x_k \) into \( y_k \) in (3) using (4), yields for \( x_0 = 0 \):
\[ y_k = qN_0(q) - N_1(q) \frac{u_k}{qD(q)} ; k \geq 1 \] (5)

with \( y_0 = 0 \) where \( q \) is the one-sample advance operator (i.e., \( qv_k = v_{k+1} \)); and
\[ N_0(q) = N^T(q)\Gamma + dD(q) \]
\[ N_1(q) = N^T(q)\Gamma_1 \]
\[ D(q) = \text{Det}(qI - \Phi) \]
\[ N^T(q) = c^T\text{Adj}(qI - \Phi) \] (6)

where \( \text{Det}(\cdot) \) and \( \text{Adj}(\cdot) \) denote the determinant and adjoint of the \( (\cdot) - \)matrix, respectively. Note that the poles of \( D(z) \) are the eigenvalues of \( \Phi(T) \), namely, \( e^{\lambda_iT} \) (\( i = 1, 2, \ldots, \nu_i \)) where \( \lambda_i \) (\( i = 1, 2, \ldots, \sigma \)) are the distinct eigenvalues of the A-matrix with multiplicities \( v_i \) (\( n = \sum_{i=1}^\sigma v_i \)). If \( \beta \neq 0 \) then there is an extra pole at \( z=0 \). The zeros of (5) are those of the polynomial
\[ N(z) = z^m(c^T\text{Adj}(zI - \Phi)\Gamma + dD(z)) + c^T\text{Adj}(zI - \Phi)\Gamma_1 \] (7)

if \( \beta \neq 0 \) and those of \( N_0(z) = 0 \) if \( \beta = 0 \) for I denoting the n-identity matrix. Since the state transition matrix of the discrete system is expanded as \( \Phi(T) = \sum_{k=0}^\infty \frac{\Lambda^k T^k}{k!} \), note from (3) that
\[ \Phi(T) = I + AT + o(T) \]
\[ \Gamma_0(T) = bT + o(T) \]
\[ \Gamma_1(\beta, T) = -\beta(I + \frac{AT}{2})b + o(T) \]
\[ \Gamma(\beta, T) = \left(1 + \frac{\beta}{T}\right)^2 + o(T^2) \]
so that the parametrization of \( N \) in \( \beta \) and \( T \) in (7) may be approximated for small \( T \) as follows:
\[ N(z, \beta, T) = c^T\text{Adj}(zI - A)\left[I + \frac{AT}{2}\right]b + d(z - 1)I - AT \] (9)

3. Limiting zeros as \( T \to 0 \)

The polynomial degree \( m = \text{deg}(c^T(sI - A)^{-1}b) \leq n-1 \), \( s \) being the Laplace operator, is the number of zeros of a strictly proper continuous plant, i.e., if the direct input-output interconnection gain is \( d = 0 \) in (1). Since \( N(z, 0, T)=N_0(z, T) \), one gets from (9) that
\[ N(z, 0, T) \to d(z - 1)^n + K(z - 1)^mT \] (11.a)

for \( \beta = 0 \) (i.e., ZOH); and
\[ N(z, \beta, T) \to (z - 1)^n \left[K_1(z - 1)^mT + K_2\right] \] (11.b)

as \( T \to 0 \) with the consistency condition \( K = c^Tb \neq 0 \), where \( \delta \beta = 1 \) for \( \beta \neq 0 \) and \( \delta \beta = 0 \), otherwise. Eq. 10 leads to

4. Features and results

**Facts 1**: Note that, if \( d=0 \), then \( c^T\text{Adj}(sI - A)b = 0 \) for a set of continuous plant zeros \( Z = \{s_i; i = 1, 2, \ldots, m \} \) if \( m \geq 1 \). No continuous zero exists if
c^T \text{Adj}(sI-A)b$ is a nonzero real constant, i.e. if $m = 0$. Thus, one has from (9) that $c^T \text{Adj}\left(\frac{z^{-1}}{T}I-A\right)\left[Tz^\delta + \left(z^\delta + 1\right)\beta\right]b = 0$ for m limiting intrinsic discrete zeros $z_i \rightarrow s_i T+1 \rightarrow 1$ ($i \in \mathbb{Z}$) as $T \rightarrow 0$ for $\beta=0$ ($\Rightarrow \delta \beta = 0$) since the change of variable $s \rightarrow z = s T+1$ is a one-to-one mapping. No intrinsic discrete zero exists if $m = 0$. One extra limiting discretization zero appears if $m \neq 0$ ($\Rightarrow \delta \beta = 1$) which is close to unity for large $|\beta|$. Such a zero depends on $\beta$ and tends to $z = \frac{K\lambda}{K(1+\lambda)}$ for $\beta = \lambda T$ being of the order of $T$ from (11.b). In particular, it tends to zero if $|\lambda|$ is very small, i.e., if $\beta \neq 0$ becomes arbitrarily faster close to zero than $T \rightarrow 0$.

Fact 2: If $d \neq 0$, then both continuous and associated discretized plants have $n$ (or $n+1$) limiting discrete zeros if $\beta=0$ (or $\beta \neq 0$). In the second case, the extra limiting zero is a discretization one.

Facts 1-2 prove, in particular, that $m$ satisfies the constraints below:

$$m = \deg \left(c^T \text{Adj} \left( zI-\Phi \right) \Gamma_o \right)$$
$$= \deg \left(c^T \text{Adj} \left( (z-1)I-AT \right) bT \right)$$
$$= \deg \left(c^T \text{Adj} \left( sI-A \right) b \right) \leq n-1 \quad (12.a)$$
$$m+1=\deg \left(c^T \text{Adj} \left( zI-\Phi \right) \left( z\Gamma + \Gamma_1 \right) \right) = c^T \text{Adj} \left( \frac{z^{-1}}{z}I-\lambda \right) \left( z + (z-1)\beta \right) b \quad (12.b)$$

for $\beta = 0$ and $\beta \neq 0$, respectively, which are also the respective degrees of the polynomials in (11) if $d=0$ (strictly proper continuous plant). If $d \neq 0$ (biproper continuous plant), then the respective polynomial degrees in (11.a) and (11.b) are $n$ and (n+1). Note from (1) and (11)-(12) that:

1) The continuous plant has no zeros if $m=d$ = $\deg \left( c^T \text{Adj} \left( sI-A \right) b \right) = 0$. The continuous plant has $n$ zeros, irrespective of the value of $m \leq n-1$, if $d \neq 0$ since $m < n = \deg \left( \text{Det} \left( sI-A \right) \right)$.  

2) $\deg \left( N \right) = n$ for all $d \neq 0$ if $\beta = 0$ (i.e., ZOH with direct input-output interconnection gain) with $n$ continuous zeros leading to $n$ intrinsic discrete ones.

3) $\deg \left( N \right) = n+1$ for all $d \neq 0$ for any given $\beta \neq 0$ with $n$ continuous zeros leading to $n$ intrinsic discrete zeros plus one extra discretization zero, namely, being related to the discretization process.

4) $\deg \left( N \right) = n-1$ if $d=\beta=0$ (i.e., ZOH with no direct interconnection gain) with $m \leq n-1$ continuous leading to $m$ intrinsic discrete zeros plus $(n-1-m)$ discretization ones.

5) $\deg \left( N \right) = n$ if $\beta \neq 0$ and $d=0$ (i.e., FROH and FOH with no direct interconnection gain) with $m$ continuous leading to $m$ intrinsic discrete zeros plus $(n-m)$ extra discretization ones which is not associated with continuous zeros.

From the above sets of numbers of discrete zeros, $m \leq n-1$ (if $d=0$) and $n$ (if $d \neq 0$) are intrinsic zeros while the remaining ($\deg(N)$-$m$) zeros are discretization ones. Note that if $d=0$ then the continuous and discrete plants are both strictly proper while if $d \neq 0$ they are biproper (but not strictly proper). Thus, the following results follow from (10)-(11): 

Results 1 (ZOH): From (11.a) and the above points 1, 2 and 4 about zeros and related degrees of polynomials, it follows for $\beta = 0$ that:

a) If $m=d=0$, i.e., the continuous plant is zero-free, then there is no limiting discrete zero located at $z=1$ as $T \rightarrow 0$ so that all limiting zeros (if any) are discretization ones.

b) If $d \neq 0$, then there are $n$ (intrinsic) discrete limiting zeros at $z=1$ as $T \rightarrow 0$ for any $m \geq 0$.

c) If $d=0$ then there are $m \leq n-1$ (intrinsic) discrete limiting zeros at $z=1$ and $(n-m)$ discretization ones as $T \rightarrow 0$. If $d$ and $T$ are arbitrarily small, with $d = \lambda T$ for some nonzero real $\lambda$, then $m$ intrinsic discrete zeros are close to unity and the $(n-m)$ remaining discretization ones are close to the roots of $K + \lambda (z-1)^{n-m} = 0$. These last ones also tend to unity if $\lambda \rightarrow \pm \infty$ (i.e., $d$ is arbitrarily large compared to $T$).

Results 2 (FROH, FOH): From (11.b) and the above points 1, 3 and 5 about zeros and related degrees of polynomials, it follows for $\beta \neq 0$ that:

a) If $d=0$ then there are $m \leq n-1$ intrinsic limiting zeros at $z=1$ as $T \rightarrow 0$ and the remaining $(n-m)$ ones are due to discretization with either any fractional or a first-order hold. Thus, there is an extra zero compared to the previously discussed case $\beta = d = 0$. The location of such a zero depends on $\beta$. In particular, it is close to zero for $\beta$ being close to zero while it tends to unity as $|\beta| \rightarrow \infty$.

b) If $d \neq 0$, then there are $(n+1)$ discrete zeros with $n$ being intrinsic discrete limiting zeros. Since $\beta \neq 0$, an extra discretization zero exists which tends to positions close to $z=1$ for large $|\beta| BT$ as $T \rightarrow 0$. From (11.b), $(m+1)$ zeros tend to $z=1$ as $T \rightarrow 0$, all of them being intrinsic limiting zeros since $(m+1) \leq n$. The remaining $(n-m)$ zeros tend to the roots of
the polynomial $K\beta + dz(z-1)^{n-m-1}$. A number $(n-m-1)$ of those zeros are intrinsic while the remaining one is a discretization zero. Such a polynomial has two roots at $z_{1,2} = \frac{d \pm \sqrt{d^2 - 4K\beta}}{2d}$ if $m = n-2$. If $m = n-1$, then there are $n=m+1$ intrinsic limiting zeros (since $d \neq 0$) which tend to $z=1$ as $T \to 0$ from (11.b) plus an extra discretization one, which tends to $z = -\frac{\beta K}{d}$ as $T \to 0$ with $z \to 0$ (i.e. to a stable cancellation with a pole) if $\left|\frac{d}{\beta}\right| \to \infty$. 

5. Example

Consider $G(s) = \frac{s+3}{(s+1)(s+2)}$ so that $d=0$. For $T = 10^{-3}$ sec., the intrinsic zero is found at $z=0.9970$ with no significant variations as $\beta$ takes any value within $[0,1]$. One extra limiting discretization zero appears if $\beta \neq 0$. As the sampling period decreases under $10^{-5}$ sec., it becomes located arbitrarily close to $z=1$ for large $|\beta|$ and arbitrarily close to $z=0$ for $\beta (\neq 0) \to 0$ as $T \to 0$.

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References


