Evolutionary Strategies for Automatic Selection of Stabilization Parameters in the Differential Game Theory

Héctor Vargas 1, Vittorio Zanella 2, Vladimir Alexandrov 3, Mónica López 4
1,2 Faculty of Computer Engineering, Universidad Popular Autónoma del Estado de Puebla, 21 Sur 1103, Colonia Santiago, Puebla, Pue., México, C.P. 72160

3 Faculty of Physical Sciences and Mathematics, Benemérita Universidad Autónoma de Puebla, Colonia San Manuel, C.U., Puebla, Pue., México, C.P. 72570

4 Faculty of Computer Science, Benemérita Universidad Autónoma de Puebla, Colonia San Manuel, C.U., Puebla, Pue., México, CP. 72570

Abstract: - In the differential game theory, problems of minimax and maximin are resolved. In these types of approaches, the minimization or maximization of a functional is carried out, which is provided by means of the criterion of a quadratic integral whose parameters are: the coordinates of the dynamic system, the control information and the defined, positive, pondering matrices. These defined, positive matrices are obtained based on the engineer’s experience on one dynamic system in particular. It is clear that the good choice of matrices, in an enclosed set, should improve the criterion’s objective. In order to automate the choice of defined positive matrices, we propose an algorithm that uses evolutionary strategies. The important result is obtained by means of two propositions which show that the operations of crossover and mutation preserve the defined, positive properties of the matrices, thereby improving the computational complexity of said algorithm.

Key-Words: - Control, differential game, evolutionary strategies, crossover operation, mutation operation, stabilization.

1 Introduction

In the theory of differential games [1], the problem of computational testing can be built and solved [2] by applying the concepts of minimax and maximin, that is, by obtaining an algorithm that evaluates and corrects the control strategy in order to stabilize a controllable, dynamic system. During the process of building the test algorithm, the expert engineer must choose, based on his experience, confidence and intuition, the positive definite matrices that obey the necessities of the dynamic system, taking into consideration its physical nature. The present article presents a way of automating the experience of the human expert by using evolutionary strategies [3], [4], [5]. The important result is achieved through two proposals which refer to the fact that the crossover and mutation operations preserve the positive definite properties of the matrices, thereby improving the computational complexity of said algorithm.

2 Problem Formulation

We consider the following ([2],[6],[7]) for the computer tests: we suppose that an unknown, control algorithm exists, from which we are only able to know its output \( u_i = u_i(t, x) \), and we furthermore want to organize a computer testing system in order to know how good this algorithm is. In other words, if we wanted to evaluate the said algorithm, how could we obtain an excellent evaluation with which we could compare the obtained evaluation with the said algorithm?.

Under the same conditions as in the minimax case [2], the maximin problem can be proposed:

\[
\inf_{u_i \in U} \mathcal{A}(x(t_i), p, u_i) \rightarrow \sup_{p \in P} x_{t_i x_i}
\]

Or its equivalent:

\[
\inf_{u_i \in U} \mathcal{A}(x(t_i), p, u_i) \rightarrow \sup_{p \in P} x_{t_i x_i}
\]

We know that the following inequality is fulfilled [8]:

\[
\sup_{p \in P} \inf_{u_i \in U} \mathcal{A}(x(t_i), p, u_i) \leq \left( \inf_{u_i \in U} \sup_{p \in P} \mathcal{A}(x(t_i), p, u_i) \right)
\]

We can therefore consider the following as an excellent grade or evaluation for the stabilization
3 Problem Solution

Within a search space, a set of points are chosen as possible solutions of the global optimal. This set is called the “initial population”, and its elements are called “population individuals”. The population individuals will be evaluated with respect to the objective function, by assigning a bonding index \( \varphi^0 \) to each individual (see Formula 4). In a series of iterations called “generations”, a portion of the population is selected by means of the selection operator. The selection criterion is a function of the bonding index. By mixing the selected subpopulation information, a new individual population is created. This population is randomly modified by means of the crossover and mutation operator, and its bonding index is evaluated.

The process is repeated until the population can not improve with the crossover and mutation, which would then be the stop condition. It is necessary to mix the information from the best individuals in order to obtain the desired optimal. The crossover and mutation operator adds new information not contained in the initial population.

Therefore, the definitions of the crossover and mutation operations will be given, and the fact that these operations preserve the properties of the positive definite matrices will be demonstrated. The section will conclude with the evolutionary algorithm.

\[ \varphi^0 = \sup_{u_1} \inf_{p} \mathcal{Z}(x(t_0), p, u_1) \quad (4) \]

where

\[ \mathcal{Z} = \int_{t_0}^{t_f} \left[ x^T G x + \Delta u^T N \Delta u \right] dt \]

\( G > 0 \), \( N > 0 \) – are positive definite matrices with \( x = A(p)x + B(p)u \), \( x(t_0) = \epsilon \), \( p \in P \).

Sometimes the algorithm will be able to obtain this excellent evaluation and sometimes it will not, depending on whether a seat point exists in the expression of the particular case being analyzed (3). Figure 1 shows how this maximin test is carried out. The control system behaves like a black box, in which we only know the input information \( tx \) and the output, which would be the control expression \( u_t \).

The inputs in the testing system are the control \( u_t \) and the system deviations \( x(t) \). This block executes the stabilization quality computer tests. It will then be possible to obtain an evaluation of the stabilization algorithm, information about the initial conditions \( x(t_0) \) and the perturbations \( p \).

These tests are organized into three stages:

**Stage 1:** Solution to the maximin problem: to obtain the excellent solution \( \mathcal{Z}^0 \); and the solution \( x^0(t_0) \), \( p^0 \) from the maximin problem, which will be used as a strategy in the second stage.

**Stage 2:** To obtain \( \mathcal{Z} \), an estimate of the real evaluation obtained by the stabilization algorithm.

**Stage 3:** Comparison of \( \mathcal{Z}^0 \) with \( \mathcal{Z} \). The closer the estimate comes to the excellent evaluation, the better the control algorithm will evaluate.

**Definition 1:** Let \( \Xi \) be the universe of symmetric, defined, positive matrices, that is,

\[ \Xi = \{ A \mid A \in \mathbb{R}^{n} \times \mathbb{R}^{n}, x^T A x > 0 \quad \forall \ x \neq 0 \quad \forall \ x \in \mathbb{R}, \ A = A^T \} \]

**Definition 2 [Crossover Operation]:** Let \( X \subset \Xi \), for all \( A, B \in X \), then \( A = U_1 \Lambda_1 U_1^T \) and \( B = U_2 \Lambda_2 U_2^T \) can be expressed where \( U_1 \) and \( U_2 \) contain unitary Eigen-vectors corresponding to their Eigen-values \( \Lambda_1 \) and \( \Lambda_2 \), respectively; therefore the crossover operation can be defined as:

\[ C_i = A \otimes B = U_i \Lambda_i U_i^T \]

where \( \Lambda_i \) represents a permutation of a subset of \( n \) elements selected from a set of \( 2n \) elements – these elements are \([\Lambda_1 \cup \Lambda_2] = 2n\), then \( i = 1, 2, \ldots, \frac{(2n)!}{(2n-n)!} \) and \( j \) alternate with each permutation carried out in the set \([1, 2]\).

**Definition 3 [Mutation Operation]:** Let \( X \subset \Xi \) for all \( A \in X \), then \( A = U \Lambda U^T \) can be expressed, therefore the mutation operation is defined as:
Proposition 1: The crossover operation conserves the properties of the positive definite matrices.

Proof: Let \( X \subseteq \Xi \), for all \( A \in X \), then \( A = U \Lambda U^T \) can be expressed, therefore the rotation \( y = U^T x \) produces the sum of the squares

\[
x^T A x = x^T U \Lambda U^T x = y^T \Lambda y
\]

where \( \Lambda = \Lambda^\prime \), then we re-assign \( A' = A \rightarrow \beta \) is the result of the mutation of its Eigen-values of the original matrix \( A \).

Proposition 2: The mutation operation conserves the properties of the positive definite matrices.

Proof: Let \( X \subseteq \Xi \), for all \( A \in X \), then all the Eigen-values of matrix \( A \) are positive, that is, \( \lambda_i > 0 \) for \( i = 1, 2, \ldots, n \), we suppose that for each \( \lambda_i > 0 \) there is a corresponding unitary Eigen-vector \( x_i \), and therefore we have \( A x_i = \lambda_i x_i \) such that

\[
x_i^T A x_i = x_i^T \lambda_i x_i = \lambda_i
\]

Since \( x_i^T x_i = 1 \), and since \( x^T A x > 0 \) \( \forall x \neq 0 \), it would be particularly valid for the Eigen-vector \( x_i \) and the quantity \( x_i^T A x_i = \lambda_i \). Now, to Equation (7) we add the \( \lambda_i \), \( \beta \in N(0,1) \), that is, \( \gamma_i = \lambda_i + \beta \), such that \( y \gamma_i > 0 \), therefore we have

\[
0 < \lambda_i + \beta_i = x_i^T (\lambda_i + \beta_i)x_i = x_i^T A' x_i
\]

In this manner, the sum of a normalized, random number \( \beta_i \in N(0,1) \), does not affect the positive, defined property of the original matrix \( A \), or the fact that the new matrix \( A' \) is positive definite. \( \square \)

The algorithm for selecting the best parameter of the optimization functional is:

STEP 1: Elements \( A \in X \subseteq \Xi \) are chosen such that \( |X| = n \) for some \( n \in N \).

STEP 2: Let \( Y \subseteq X \) such that \( \forall A \in Y \) is fulfilled such that

\[
\Im(A) \leq \Im(B) \quad \forall B \in X,
\]

and that \( \chi = \min \{ \Im(A) / A \in Y \} \).

STEP 3: Let \( Y = W \cup Z \) with \( W \cap Z = \Phi \) and we carry out the operations of crossover and mutation over the elements of \( W \) and \( Z \), respectively, in order to obtain new subsets:

\[
W' = \{ C_i / C_i = A \otimes B \quad \forall A, B \in W \}, \quad Z' = \{ D / D = A \rightarrow \sigma \quad \forall A \in Z \text{ with } \sigma \in N(0,1) \}
\]

STEP 4: Let \( Y' = W' \cup Z' \) and we obtain

\[
\chi' = \min \{ \Im(A) / A \in Y' \}.
\]

STEP 5: If \( \chi' - \chi \leq 0 \) then we re-assign \( X = Y' \) and return to STEP 2.

If the opposite occurs, the new generation does not improve the population quality, therefore a stronger mutation can be done, or we can finish the evolution if \( \chi' \approx 0 \).

Thus, the choice of the functional’s parameter \( \Im(A) \), the positive, defined matrix \( A \), is such that \( \chi' = \Im(A) \) for some \( A \in Y' \) such that \( \chi' \approx 0 \).

4 Results

We consider the following mathematical model:

\[
\dot{x}_i = u_i, \quad u_i = -k x_i - k_x \dot{x}_i
\]

We wish to solve the problem

\[
\max_{|t|_{0} = \ldots} \int_{0}^{\infty} (x_i^2 + \dot{x}_i^2 + \ddot{x}_i^2) dt \to \min_{|u|_{0} = \ldots}
\]

whose physical sense is the following: given the worst initial conditions, we want to minimize the system’s deviation, as well as the velocity and acceleration of these deviations.

In this case

\[
A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad K^T = (k_x, k_x),
\]

\[
x^T = (x_i, \dot{x}_i)
\]
The expert specialist chooses the following positive, defined matrix

\[
G^T = G = \begin{pmatrix} 0.2 & 0.1 \\ 0.1 & 0.2 \end{pmatrix}, \quad n_0 = 1
\]

In this example, the set point exists, with is fulfilled such that

\[
\min_{k \in Q} \max_{|x(t_0)| = 0} \max_{|x(t)| = 0} \min_{k \in Q} \mathcal{J} = \max_{k \in Q} \min_{|x(t_0)| = 0} \min_{|x(t)| = 0} \mathcal{J} \quad \text{where}
\]

\[
\mathcal{J} = \int_{t_0}^{t} (x_i^2 + \dot{x}_i^2 + \ddot{x}_i^2) dt
\]

After verifying that System (9) is completely controllable, we proceed to the calculation of internal Problem [2], that is:

\[
\min_{k \in Q} \mathcal{J} = \min_{k \in Q} \int_{t_0}^{t} (x_i^2 + \dot{x}_i^2 + \ddot{x}_i^2) dt = x^0^T(t_0) L_0 x^0(t_0)
\]

Therefore, the external problem \( \max_{|x(t_0)| = 0} x^T(t_0) L_0 x(t_0) = \mu_{\max} \) is solved. Thus, an optimum value for the functional \( 1 + \sqrt{3} \) was obtained for the worst initial conditions. The graph is shown below.

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**Fig 2.** Asyntotic behavior of the solution when the positive, defined matrix \( G \) has been fixed, chosen by a human specialist.

Now we present the result of applying the algorithm based on evolutionary strategies, in order to automatically choose the best, positive definite matrix \( G \) without the intervention of a human expert. The following graph (fig. 3) presents the behavior of the bonding index.

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**Fig 3:** Results of the bonding index in each iteration step.

When the functional does not change in the next generations, this means that we have reached the best individual, that is, the best, positive definite matrix \( G \), which, when applied to the functional, reduces our stabilization time, as is shown in the following graph.

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**Fig 4:** Results of the behavior of the system’s coordinates, for the best choice of the pondering matrix, given the worst initial condition.

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5 Conclusion

The two proposals demonstrate that the crossing and mutation operations preserve the structures of the positive definite matrices. This means that within each iteration of the algorithm, there is no need to verify whether the new generation belongs to its species or not. In other words, the new individual generated by either a mutation or a crossing has the characteristics of being a positive definite matrix. This avoids increasing the complexity in time of the algorithm based on evolutionary strategies.

The algorithm based on evolutionary strategies resulted in improving the choice made by the human
specialist, thereby obtaining complete autonomy of the computer tests algorithm. Even though it has been proven by means of an example, we can extend its use to include more robust systems by means of the conception and construction of said algorithm. By comparing Graphs 2 and 4, one can observe that the stability of the dynamic system improves.

References: