On Normed Space of Ordered Fuzzy Numbers

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Abstract

To make calculation with fuzzy numbers more efficient and similar to the real (crisp) number calculus, the concept of the membership function of a fuzzy set, introduced by L. Zadeh in 1965, is weakened by requiring a mere *membership relation*. In this way a generalized concept of a fuzzy number appears as an ordered pair of continuous real functions defined on the interval $[0, 1]$. In this way the graph of the membership relation is equipped with its orientation. Moreover, to a given ordered fuzzy number two kinds of opposite elements are defined: the classical opposite element obtained from the number by its multiplication with a negative crisp one, and the complementary element which differs from the opposite one by the orientation. Four algebraic operations between such fuzzy numbers are constructed in a way that renders them an algebra. Further, a normed topology is introduced which makes them a Banach space, and even more, a Banach algebra with unity. In the particular case the operations on the so-called convex fuzzy numbers can be recovered.

Keywords: fuzzy number, fuzzy membership relation, algebraic operations, normed space, algebra

1 Introduction

The commonly accepted model of calculations on fuzzy numbers is that set up by Duboisia and Prade [3], who proposed a restricted class of membership functions, called (L, R)–numbers. The essence of their representation is that the membership function is of a particular form that is generated by two so-called shape functions: L and R. If functions L and R are linear, the membership functions of fuzzy numbers become triangular. However, approximations of fuzzy functions and operations are needed, if one wants to stay within this representation while following the Zadeh's extension principle [2]. It leads to some drawbacks concerned with the properties of fuzzy algebraic operations, as well as to unexpected and uncontrollable results of repeatedly applied operations, caused by the need of intermediate approximations [12].

One of the goals of our paper is to construct a revised concept of a fuzzy number, and at the same time to have the algebra of crisp (non-fuzzy) numbers inside the concept. The other goal is to preserve as much of the properties of the classical so-called *crisp reals* $\mathbb{R}$ as possible, in order to facilitate real world applications as e.g. in fuzzy control systems. The new concept makes possible utilizing the fuzzy arithmetic and constructing an algebra of fuzzy numbers. By doing this, the new model of fuzzy numbers has obtained an extra feature, which was not present in the previous ones: neither in the classical Zadeh's model, nor in the more recent model of so-called convex fuzzy numbers. This feature, called in [17, 18, 20] the orientation, requires a new interpretation as well as a special care in dealing with ordered fuzzy numbers. To avoid confusion at this stage of development, let us stress that any fuzzy number, classical (crisp or con-

vex fuzzy) or ordered (new type), has its *opposite number*, which is obtained from the given number by multiplication with minus one. For the new type of fuzzy numbers, multiplication by a negative real not only affects the support, but also the orientation swaps. Important is that to a given ordered fuzzy number two kinds of opposite elements are defined: the classical, one can say an algebraic opposite number(element) obtained by its multiplication with a negative crisp one, and the complementary number which differs from the opposite one by the orientation. Relating to an ordered fuzzy number its opposite and complementary elements makes the calculation more complex, however with new features. From one side a sum of an ordered fuzzy number and its algebraic opposite gives a crisp zero, like in the standard algebra of real number. From another side the complementary number can play the role of the opposite number in the sense of the Zadeh's model, since the sum of the both – the (ordered fuzzy) number and its complementary one – gives a fuzzy zero, non-crisp, in general. We have to admit that the applicability of the new type of fuzzy numbers is restricted to such real-life situations, where also the modelled circumstances provide information about orientation. In particular, in most existing approaches, for a fuzzy number a the difference $a - a$ gives a fuzzy zero. However, this leads to unbounded growth of the support of fuzziness if a sequence arithmetic operations is performed between two (classical) fuzzy number. To overcome this unpleasant circumstance the concept of the orientation of a fuzzy number has been introduced as well as simple operations between those new objects, called here ordered fuzzy numbers, which are represented by pairs of continuous functions defined on the unit interval[0,1]. Those pairs are the counterparts of the inverses of the increasing and decreasing parts of convex fuzzy numbers. In particular case, for the pairs (f, g) where $f, g \in C^0([0, 1])$ which satisfy:
1)$f \leq g$ and 2)f and g are invertible, with f increasing and g decreasing, one can recover the class of fuzzy numbers called convex ones [4, 11]. Then as long as multiplication by negative numbers are not performed classical fuzzy calculus is equivalent to the present operations defined for ordered fuzzy numbers (with negative orientation).

Doing the present development, we would like to refer to one of the very first representations of a fuzzy set defined on a universe X (the real axis $\mathbb{R}$, say) of discourse. In that representation (cf. [1, 5]) a fuzzy set (read here: a fuzzy number) A is defined as a set of ordered pairs $\{(x, \mu_x)\}$, where $x \in X$ and $\mu_x \in [0, 1]$ has been called the grade (or level) of membership of x in A. At that stage, no other assumptions concerning μ_x have been made. Later on, one assumed that μ_x is (or must be) a function of x. However, originally, A was just a relation in a product space $X \times [0, 1]$.

2 Attempts

A number of attempts to introduce non-standard operations on fuzzy numbers have been made [8, 7, 10, 11, 12]. It was noticed that in order to construct operations more suitable for their algorithmisation a kind of invertibility of their membership functions is required. In [4, 15, 13, 6] the idea of modelling fuzzy numbers by means of convex or quasi-convex functions (cf. [14]) is discussed. We continue this work by defining quasi-convex functions related to fuzzy numbers in a more general fashion, enabling modelling both dynamics of changes of fuzzy membership levels and the domain of fuzzy real itself. Even starting from the most popular trapezoidal membership functions, algebraic operations can lead outside this family, towards such generalized quasi-convex functions.

That more general definition enables to cope with several drawbacks. Moreover, it seems to provide a solution for other problems, like, e.g., the problem of defining total ordering over fuzzy numbers (cf. [18]). Here we should mention that Klir was the first, who in [10] has revised fuzzy arithmetic to take relevant requisite constraint (the equality constraint, exactly) into account and obtained $A - A = 0$ as well as the existence of inverse fuzzy numbers for the arithmetic operations. Some partial results of the similar importance were obtained by Stanch in [7] by introducing an extended operation of a very complex structure. Our approach, however, is much simpler from mathematical point of view, since it does not use the extension principle but refers to the functional representation of fuzzy numbers in a more direct way.

In the classical approach the extension principle which gives a formal apparatus to carry over operations (e.g. arithmetic or algebraic) from sets to fuzzy sets. Then in the case of the so-called convex fuzzy numbers (cf. [11] the arithmetic operations are algorithmically with the help of the so-called α-sections of membership functions. The local invertibility of quasi-concave membership functions, on the other hand, enables to define operations in terms of the inverses of the corresponding monotonic parts, as was pointed out in our previous papers [15, 16, 17, 18]. In our last paper [20] we went further and have defined a more general class of fuzzy number, called ordered fuzzy number, just as a pair of continuous functions defined on the interval $[0, 1]$. Those pairs are counterparts of the mentioned inverses.

3 Ordered fuzzy numbers

Here the concept of membership functions is weakened by requiring a mere a particular type of a *membership relation* or multifunction.

Definition *By an ordered fuzzy number* $a \in \mathcal{R}$ *we mean an ordered pair of functions*

$$a = (f, g) \tag{1}$$

where elements of the pair are continuous functions $f, g : [0, 1] \rightarrow \boldsymbol{R}$. We call the corresponding elements: f – an **up-part** of a and then denote by a_{up}, and g – the **down-part** of the fuzzy number a then denote by a_{down}.

It was pointed out in [17], to make the set of ordered fuzzy numbers closed under arithmetic operations, the assumption the up-branch comes before the down-branch (which is the case of convex fuzzy numbers),has to be dropped. Graphically the curves (f, g) and (g, f) do not differ, if drawn on the coordinate system in which x-axis proceeds y axis. However, the corresponding curves determine two different ordered fuzzy numbers: they differ by the orientation: if the first curve has the positive orientation, then the second one has negative. It will be seen in the figure below.

According to the definition introduced in our previous papers (cf. [20] and [19] we perform arithmetic operations componentwise. In the present notation they will be

$$\begin{aligned} (a+b)_{up} &= a_{up} + b_{up} \qquad (2) \\ (a+b)_{down} &= a_{down} + b_{down} \\ (a-b)_{up} &= a_{up} - b_{up} \\ (a-b)_{down} &= a_{down} - b_{down} \\ (a \cdot b)_{up} &= a_{up} \cdot b_{up} \qquad (3) \\ (a \cdot b)_{down} &= a_{down} \cdot b_{down} \\ (a/b)_{up} &= a_{up}/b_{up} \\ (a/b)_{down} &= a_{down}/b_{down} \end{aligned}$$

where the division is only defined when 0 does not belong to the values of b_{up} and b_{down}. In the particular case when a subset of those numbers for which the up-branch comes before the down-branch the operations on the so-called convex fuzzy numbers can be recovered.

4 Normed space of fuzzy numbers

Let us notice that all operations defined are suitable for pairs of functions. The pointwise multiplication has a neutral element – the pair of two constant functions equal to one.

Linear structure of $\mathcal{R}$ is obvious: the set of all pairs of continuous functions $\mathcal{R}$ is isomorphic to the linear space of real 2D vector-valued functions defined on the unit interval $I = [0, 1]$.

Normed structure of $\mathcal{R}$ is introduced by the norm:

$$||a|| = \max(\sup_{s \in I} |a_{up}(s)|, \sup_{s \in I} |a_{down}(s)|).$$

Hence $\mathcal{R}$ can be identified with $C([0, 1]) \times C([0, 1])$.

The space $\mathcal{R}$ is an Abelian group and topologically a Banach space, and moreover, a **Banach algebra** with the unity $(1, 1)$ - a pair of constant functions equal to one. Possessing the space $\mathcal{R}$ equal to $C^0([0,1]) \times C^0([0,1])$ gives us the chance to define a large set of defuzzyfication procedures of ordered fuzzy numbers [21] due to the general representation theorem (of Banach-Kakutami-Riesz) for linear and continuous functionals; they are uniquely determined by a pair of Radon measures μ on $[0,1]$

A partial order in the set $\mathcal{R}$ can be introduced by defining the subset of 'positive'ordered fuzzy numbers. We say the fuzzy number pair $a = (a_{up}, a_{down})$ is not less than zero, and write

$$a \geq 0 \quad \text{iff} \quad (a_{up} + a_{down}) \geq 0, \tag{4}$$

where the plus is taken pointwise .

We should notice, that after publishing our recent paper [20] we were told about the paper of Goetschel and Voxman [9], in which a Banach structure of an extension of convex fuzzy numbers was introduced. However, the authors of [9] were only interested in the linear structure of this extension.

5 Examples

In this section we define some fuzzy numbers and perform the previously defined operations of them.

For reference, in the following Tab. 1 we give the formulas by which the plotted examples have been generated.

$$\begin{aligned} a_{up} &= 1 - (id - 1)^2 \\ a_{down} &= 1 + (1 - id)^2 \\ b_{up} &= 2 + id \\ b_{down} &= 4 - id \\ c_{up} &= 4 - id \\ c_{down} &= 2 + id \end{aligned}$$

Table 1: Definitions of some fuzzy numbers.

In all figures plots of each pair are made in accordance to the classical representation, where y variable(the vertical one) is regarded as a function of x variable (the horizontal one).

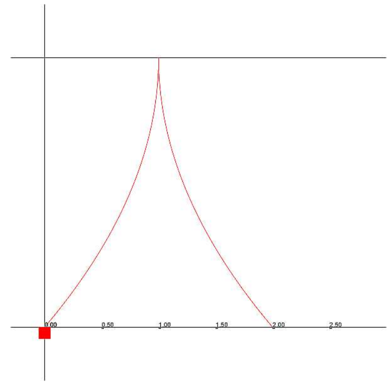

Fig. 1: Ordered fuzzy number a .

Moreover, the first element of the pair contains a small square at the bottom, when the x axis is reached. Here the function id denotes the identity, i.e. $id(y) = y$ for any $y \in [0,1]$, while the square is taken pointwise (not superposition).

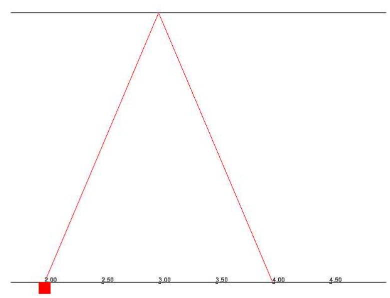

Fig. 2: Ordered fuzzy number b.

Notice that the numbers b and c differ in the orientation, only: the number c has positive orientation (with respect to the assumed coordinate system), while the number b has negative one. One can say that the ordered number c is the reverse of the number b. In next figures results of operations are presented.

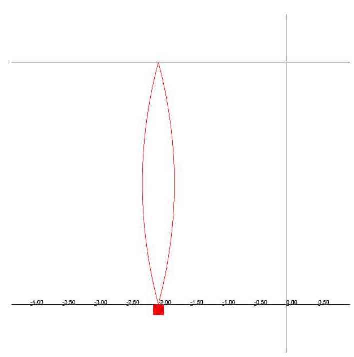

Fig. 3: Difference d of a and b.

The final pair of pictures represent the result of two other algebraic operations.

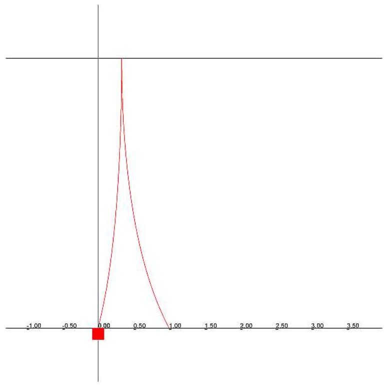

Fig. 4: Ratio q of a and b.

Notice the number d is not-proper: its membership relation is not functional.
At this point it could be suitable to refer to our previous papers (cf. [17, 18, 19, 20] and to recall previous definition and denotations:
Definition 2(after [20]) *By* **ordered fuzzy real** *we mean ordered pair* $A = (\mu_A^{\uparrow}, \mu_A^{\downarrow})$, *where* $\mu_A^{\uparrow}, \mu_A^{\downarrow} : [0, 1] \to \mathbb{R}$ *are continuous functions.*

Notice that here arrow up and arrow down are use for up-branch and down-branch (down-part), respectively.
In [18], a new concept of a fuzzy observation $\boldsymbol{f}$, has been introduced, which can help in finding appropriate motivations for the concept introduced. However, because of the lack of the space we are not going to develop this concept, and refer the reader to that paper. Notice, that the fuzzy observation can play the role of the primitive concept in this approach.

6 Conclusions

Existing algebraic operations on fuzzy numbers, especially those for fuzzy numbers of L–R type or convex fuzzy numbers (see [11]) and those basing on the Zadeh's extension principle are leading to interval analysis and several drawbacks: i) they do not possess neutral elements of addition and multiplication, ii) they lead to the blow-up of the width of supports (i.e. fuzziness) after multiple fuzzy operations, iii) they cannot be equipped with a linear structure and hence any norm.
In our opinion all those drawbacks are eliminated in the new model. Moreover, in the new model the fuzzy numbers the extra feature, called the orientation requires a new interpretation as well as a special care in dealing with ordered fuzzy numbers. For the new type of fuzzy numbers one can relate additionally to its opposite also its *complementary number*, which can play the role of the opposite number in the sense of the Zadeh's model, since the sum of the both – the (ordered fuzzy) number and its complementary number – gives a fuzzy zero, non-crisp in general. However, the sum of any new type fuzzy number and its opposite number gives the crisp zero. This is due to the fact that multiplication by negative scalars, i.e. crisp negative numbers, affects not only the support but also the orientation.
The proposed operations have been implemented in the form of a *fuzzy calculator* working as the algebra under Windows by Mr.Roman Koleśnik. First, however, a fuzzy arithmometer was implemented in the Delphi environment by my Ph.D. student Mr.Piotr Prokopowicz. [16, 17, 19].
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