CNN Models of Nonlinear PDEs with Memory

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Abstract:-In this paper a parabolic equation with memory operator is considered. CNN model for such equation is made. Dynamic behavior of the CNN model is studied using describing function method. Traveling wave solutions are proved for the CNN model. An example of one-dimensional wave in medium with memory arising in classical mechanics is presented.
Key-Words:- Cellular Neural Networks, Partial Differential Equations, Hysterezis

1. Introduction

The main aim of this paper is to study a class of first order parabolic equation, in which a memory operator occurs in the sourse term [5]:

$$\frac{\partial u}{\partial t} - \Delta u + \mathcal{F}(u) = 0, \text{ in } Q, \quad (1)$$

where we assume that $\Omega \subset \mathbf{R}^N$, $(N \geq 1)$ is an open set of Lipschitz class, fix $T > 0$ and set $Q := \Omega \times]0, T[$. The unknown u may represent the temperature and $\mathcal{F}(u)$- a space distribution of thermostats, characterized by continuous hysteresis cycles.

We will search for traveling wave solutions of such model which leads to study of ordinary differential equations with hysteresis. In this connection we will construct CNN model of (1) and we will study its dynamical behavior using describing function method. Finally we will present a CNN model of one- dimensional wave in medium with memory arising in classical mechanics.

2. CNN Model for Equation with Memory

For solving the parabolic equation with memory (1) spatial discretization has to be applied. The partial differential equation is transformed into a system of ordinary differential equations which is identified as the state equations of a CNN with appropriate templates.

There are several ways to approximate the Laplacian operator Δ in discrete space by a CNN synaptic law with an appropriate A-template [4]. For example we can have:

a). one-dimensional discretized Laplacian template

$$A_1 : (1, -2, 1); \quad (2)$$

b). two-dimensional discretized Laplacian template:

$$A_2 : \begin{pmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad (3)$$

Let us consider an autonomous CNN with $N \times N$ cells lined up in a row and let compare (2) with the state equation of the autonomous CNN. Then we obtain the following templates:

$$\begin{aligned} &A = [1, -2, 1] \quad (4) \\ &\tilde{A} = [0, \ \ -\mathcal{F}(u_j), \ \ , 0], \\ &1 \leq j \leq M = N.N. \end{aligned}$$

We will take the memory operator $\mathcal{F}(u_j)$ to be a real hysteresis functional defined by an "upper" function $\mathcal{F}_U$ and a "lower" function $\mathcal{F}_L$ (Fig.1). Functions $\mathcal{F}_U$ and $\mathcal{F}_L$ are real valued, piecewise continuous, differentiable functions. Moreover, $h(v_{xij})$ is odd in the sense that

$$\mathcal{F}_U(u_j) = -\mathcal{F}_L(-u_j).$$

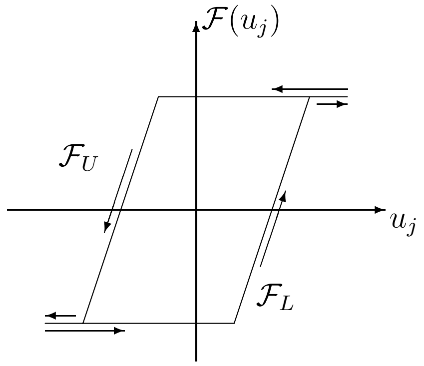

Figure 1: Hysteresis nonlinearity

For the output function f of our model we will take the standard sigmoid function. We will take periodic boundary conditions:

$$\begin{aligned} u_0 &= u_M, \quad (5) \\ u_{M+1} &= u_1, \end{aligned}$$

which make the array circular [4].

3. Dynamic Behavior of the CNN Model

Let us take for simplicity the following hysteresis functional $\mathcal{F}(u_j) = \frac{u_j^3}{3} - u_j$. Then our CNN model can be written in the following form:

$$\frac{du_j}{dt} = u_{j-1} - 2u_j + u_{j+1} - (\frac{u_j^3}{3} - u_j), \quad (6)$$
$$1 \leq j \leq M = N.N$$

or

$$\frac{du_j}{dt} = u_{j-1} - u_j - u_{j+1} + n(u_j), \quad (7)$$

where the nonlinearity is $n(u_j) = -\frac{u_j^3}{3}$.

In this paper we investigate the dynamic behavior of a CNN model (7) by use of Harmonic Balance Method well known in control theory and in the study of electronic oscillators [3] as describing function method. The method is based on the fact that all cells in CNN are identical [1], and therefore by introducing a suitable double transform, the network can be reduced to a scalar Lur'e scheme [3].

We are looking now for possible periodic solutions of the system (7) of the form.

$$U_{\Omega_0}(\omega_0) = U_{m_0} sin(\omega_0 t + j\Omega_0). \quad (8)$$

Then we can approximate the output in the same way:

$$V_{\Omega_0}(\omega_0) = V_{m_0} sin(\omega_0 t + j\Omega_0).$$

According to the describing function method we take the first harmonics, i.e. $j = 0 \Rightarrow$

$$U_{\Omega_0}(\omega_0) = U_{m_0} sin\omega_0 t, \qquad (9)$$

$$V_{\Omega_0}(\omega_0) = V_{m_0} sin\omega_0 t, \qquad (10)$$

and we can find the amplitude V_{m_0} of the output:

$$V_{m_0} = \frac{1}{\pi}\int_{-\pi}^{\pi} N(U_{m_0} sin\psi) sin\psi \, d\psi = \qquad (11)$$
$$= -\frac{U_{m_0}^3}{4}.$$

Suppose that our CNN model (7) is a finite circular array of M cells. For this case we have finite set of frequences:

$$\Omega_0 = \frac{2\pi k}{M}, \ \ 0 \leq k \leq M-1. \qquad (12)$$

According to the describing function method [3], if for a given value of Ω_0 from (12) we can find ω_0 and U_{m_0}, then we can predict the existence of periodic solution of our CNN model for the parabolic equation (1). Therefore, we have:

Proposition 1 *CNN model (7), with circular array of $M = N.N$ cells and periodic boundary conditions*

$$u_0(t) \equiv u_M(t),$$

$$u_{M+1} \equiv u_1(t),$$

has periodic solution with period $T_0 = 2\pi/\omega_0$ and amplitude U_{m_0} for all $\Omega_0 = \frac{2\pi k}{M}$, $0 \leq k \leq M-1$.

Remark 1. According to the Poincare-Bendixon theorem [3] applied to our case, only a set of initial conditions of measure zero will reach a periodic solution, all other trajectories will converge to an equilibrium point.

Remark 2. (Regulazing effect of hysteresis) Proposition 1 shows that the presence of hysteresis has a regulazing effect in nonlinear wave propagation, in essential contrast to the possible occurance of discontinuous solutions in the form of shock waves that can develop for the nonlinear wave equation, that is, in the case of nonlinear superposition operator.

As we said in the begining we will search for traveling wave solution of (1). We look for a solution in the form: $u(x,t) = \hat{u}(x+ct)$, where c is the speed of the wave. It is known [5] that for a traveling wave front represented by $u(x,t)$ is said to be a wave front if

$$u(x,t) \rightarrow k_1 \text{ as } t \rightarrow -\infty,$$

$$u(x,t) \rightarrow k_2 \text{ as } t \rightarrow \infty,$$

for some constants k_1 and k_2.

According to the obtained results, there exist stable periodic solutions of our CNN model (7) and such that $lim_{t\rightarrow \pm\infty} u_j(t) = const.$, $1 \leq j \leq M$. Therefore we have proved existence of traveling wave solutions with period $T_0 = 2\pi/\omega_0$ and the wave front U_{m_0}.

4. Nonlinear Waves in Medium with Memory

This section deals with one-dimensional waves in medium with memory. Following [2] we shall denote by x a co-ordinate of a point belonging to a solid body, by t- the time variable, by ε- the deformation, by σ- the tension and b

$$\varepsilon(t) = \int_{-\infty}^{t} \sqrt{1+K^*}\sqrt{a'(\sigma)} \qquad (13)$$
$$\sqrt{1+K^*}\sqrt{a'(\sigma)}\sigma_t' \, dt.$$

In the previous equality K^* is the convolution operator:

$$K^* u(t) = \int_{-\infty}^{t} K(t-\tau)u(\tau)\, d\tau, \qquad (14)$$

$\sqrt{1+K^*}$ stands for the development of the operator $1+K^*$ into a power series and the integral operator $\sqrt{1+K^*}$ as well as the multiplication operator $a^{'}(\sigma)$ are acting on the function $\sigma_t^{'}$.

It is well known from classical mechanics that the next equation holds:

$$\frac{\partial^2 \varepsilon}{\partial t^2} - \frac{\partial^2 \sigma}{\partial x^2} = 0, \qquad (15)$$

supposing ε and σ to be smooth functions of (t,x).

Putting $(\sqrt{1+K^*})^{-1} = 1 - \Phi^*$,

$$\Phi^* u = \int_{-\infty}^{t} \Phi(t-\tau) u(\tau)\, d\tau$$

we see that each smooth solution σ of the nonlinear integro-differential equation

$$\sqrt{a'(\sigma)} \frac{\partial \sigma}{\partial t} \pm (1-\Phi^*) \frac{\partial \sigma}{\partial x} = 0, \;\; \sigma \in C^2 \qquad (16)$$

$$(x \geq 0)$$

will satisfy (15) with ε given by (13).

According to the mechanical terminology the function Φ is called "kernel of heredity". Assume that

$$\Phi(t) = k e^{-kt}, \;\; k > 0.$$

So we have that a wave of tension, propagating "to the right-hand side" is given by next nonlinear first order equation:

$$\frac{\partial}{\partial t} \int_0^{\sigma(t,x)} \sqrt{a'(\lambda)}\, d\lambda + \frac{\partial \sigma}{\partial x} + \qquad (17)$$

$$k \int_0^{\sigma(t,x)} \sqrt{a'(\lambda)}\, d\lambda) = 0,$$

$$\sigma(t,0) = \sigma_0(t),\; \sigma_0(t) \equiv 0,\; t \leq 0,\; \sigma = 0,$$

$$\text{for } x \geq 0, t \leq 0, \sigma_0 \in C^2(R).$$

Let us make the change of the unknown function

$$w = \int_0^{\sigma} \sqrt{a'(\lambda)}\, d\lambda. \qquad (18)$$

Obviously, $w^{'} = \sqrt{a^{'}} > 0 \Rightarrow$ there exists $\varphi \in C^2$, such that

$$\sigma = \varphi(w). \qquad (19)$$

Then (17) will be rewritten in the form

$$\frac{\partial w}{\partial t} + \varphi^{'}(w) \frac{\partial w}{\partial x} + kw = 0, \quad (20)$$

$$w(t,0) = \int_0^{\sigma_0(t)} \sqrt{a'(\lambda)}\, d\lambda \equiv w_0(t),$$

$w_0(t) \equiv 0$ for $t \leq 0$, $w_0 \in C^2(R)$, $w = 0$ for $x \geq 0$, $t \leq 0$.

Let us consider equation (20) in the following form:

$$\frac{\partial w}{\partial t} = -\varphi^{'}(w) \frac{\partial w}{\partial x} - kw. \qquad (21)$$

We map $w(x,t)$ into a CNN layer such that the state voltage of a CNN cell $x_{ij}(t)$ at a grid point (i,j) is associated with $w(ih,t)$, $h = \Delta x$. Hence, the following CNN model is obtained:

$$\frac{dw_i}{dt} = -\varphi^{'}(w_i) \frac{(w_{i+1} - w_{i-1})}{h} - kw_i. \qquad (22)$$

If we compare the above equation with the state equation of nonlinear CNN we directly find the templates:

$$\hat{A} = [\frac{\varphi^{'}}{h} \quad -k \quad -\frac{\varphi^{'}}{h}].$$

We will consider the following examples for our CNN model (22):

Let $a(\lambda) = \frac{e^{2\lambda}}{2} - \frac{1}{2}$. Then $w = \int_0^{\sigma} \sqrt{a^{'}} d\lambda = e^{\sigma} - 1 \Rightarrow \sigma = ln(w+1)$ and $\varphi^{'}(w) = \frac{1}{w+1}$. The initial condition is:

$$w_0 = \begin{cases} 0, & t \leq 0, \\ 1 - cos\, t, & t > 0. \end{cases}$$

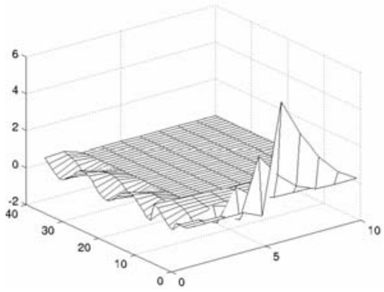

Figure 2: Nonlinear wave model

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