Periodic Solution and Global Robust Exponential Stability of Interval Cellular Neural Networks with Time-varying Delays *

Anping Chen[1], Jinde Cao[2]

1. Department of Mathematics, Xiangnan University,

Chenzhou, 423000, Hunan, P. R. China

anping_chen@hotmail.com; chenap@263.net

2. Department of Mathematics, Southeast University,

Nanjing, Jiangsu 210096, China.

Abstract

For a class of neural dynamic system with time-varying perturbations in the time-delayed state, this article studies the periodic solution and global robust exponential stability. A series of new criterions concerning the existence of the periodic solution and global robust exponential stability are obtained by employing the Young's inequality, Lyapunov functional and combine with some analysis techniques. At the same time, the global exponential stability of the equilibrium point of the system is also obtained. Several previous results are improved and generalized. Compared with existing results, our results are shown to be more effective than other ones. In addition, these results can be used to design globally stable neural networks and periodic oscillatory neural networks, and they are easy to be checked and be applied in practice.

1. Introduction

In recent years, there has been increasing interest in study of neural networks since they have a wide range of applications, for example, pattern recognition, associative memory, and combinatorial optimization. These applications heavily depend on the dynamical behaviors of the neural networks. Specially, cellular neural networks (CNNs) has attracted the attention of scientific researchers [see 1-17]. The

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circuit diagram and connection pattern modelling a CNN can be found in Refs.[1, 2]. It is well known that, In the implementation of artificial neural networks, due to the finite switching speed of amplifies, time delays are unavoidably encountered[3]. time-delayed perturbations may induce system instability, oscillation or degraded performances. On the other hand, during the implementation on very-scale-integration(VLSI) chips, external disturbance and parameters perturbation can also destroy the stability of the neural networks. Very recently, Liao[4] has investigated the robust stability of DCNNs with constant coefficient perturbations and constant delays and obtained some important results. However, to make the result applicable, This condition that the time-delay $\tau_{ij}(t)$ is time-invariant obviously limits the use of the result, since these perturbations and time-delays existing in most practical engineering systems are seldom time-varying bounded function and time-delays satisfying $\tau_{ij}^{'}(t) < 1$. Based on the neural networks system with these uncertainties, the study of the stability of the cellular neural networks with time-varying perturbations and time-varying delays possess an important significations in practice. the problem of obtaining condition of robust stability is of great theoretical and practical interest. Therefore, the main purpose of this paper is to study periodic solution and global robust exponential stability(GRES) for a class of cellular neural networks with time-varying delays and time-varying perturbations, where time-varying perturbations and time-varying delays are not periodic, only the control input $I_i(t)$ is periodic. As a spacial case, when the control input item $I_i(t)$ and time-varying perturbation item are time-invariant, we obviously obtain the global robust exponential stability of the equilibrium point of the system. These results improve and generalize those given in the earlier references[4-8].

The organization of this paper is as follows. In section 2, we give some preliminaries. In section 3, we state our main results and proofs. In section 4, we shall show an example to illustrate effectiveness of our main results. In section 5, we give some concluding of the results.

2. Preliminaries

In the section, we state our problem, definitions and some lemmas.

We consider the robust stability of the following CNNs with time-varying delays

$$\left\{\begin{array}{rcl} x_i'(t) &=& -a_i(t)x_i(t) + \sum\limits_{j=1}^{n} w_{ij}(t)f_j(x_j(t)) + \sum\limits_{j=1}^{n} w_{ij}^{\tau}(t)g_j(x_j(t-\tau_{ij}(t))) + I_i(t), \\ A_+ &=& \{A(t) = \mathrm{diag}(a_i(t))_{n\times n}: \quad \underline{A} \le A(t) \le \overline{A}, \quad i.\ e. \quad \underline{a}_i \le a_i(t) \le \overline{a}_i\}, \\ W_+ &=& \{W(t) = (w_{ij}(t))_{n\times n}: \ \underline{W} \le W(t) \le \overline{W},\ i.\ e.\ \underline{w}_{ij} \le w_{ij}(t) \le \overline{w}_{ij}\}, \\ W_+^{\tau} &=& \{W^{\tau}(t) = (w_{ij}^{\tau}(t))_{n\times n}: \ \underline{W}^{\tau} \le W^{\tau}(t) \le \overline{W}^{\tau},\ i.\ e.\ \underline{w}_{ij}^{\tau} \le w_{ij}^{\tau}(t) \le \overline{w}_{ij}^{\tau}\}, \end{array}\right. \tag{1}$$

where $i, j = 1, 2, \cdots, n$. $\underline{A}, \overline{A}, \underline{W}, \overline{W}, \underline{W}^{\tau}, \overline{W}^{\tau}$ are all constant matrix, and $\underline{A} > 0$; $0 < \tau_{ij}(t) < \tau$ and $\tau_{ij}^{'}(t) \le \delta_{ij} < 1$; $f_j(x_j(t))$ and $g_j(x_j(t-\tau_{ij}(t)))$ denote the output of the j-th unit at time t and $t-\tau_{ij}(t)$, respectively; Only $I_i(t)$ is a $\omega-$ periodic function, i.e. $I_i(t+\omega)=I_i(t)$, for any $t\in\mathbf{R}$.

The initial condition of system (1) is as follows

$$x_i(s)=\varphi_i(s),\ \ s\in[-\tau,0]. \tag{2}$$

where $\varphi_i\in C([-\tau,0],\mathbf{R})$, $C([-\tau,0],\mathbf{R}^n)$ be the Banach space of continuous function which map $[-\tau,0]$ into $\mathbf{R}^n$ with topology of uniform converge.

We denote the solution of system (1) through $(0,\varphi)$ as

$$x(t,\varphi)=(x_1(t,\varphi),x_2(t,\varphi),\cdots,x_n(t,\varphi)).$$

Set

$$x_t(\varphi)=x(t+\theta,\varphi),\quad \theta\in[-\tau,0],\quad t\geq 0.$$

then, we have $x_t(\varphi)\in C$, $\forall t\geq 0$.

For any $x(t)\in C([-\tau,0],\mathbf{R}^n)$, we define its norm as

$$\|x\|=\sup_{-\tau\leq t\leq 0}\left(\sum_{i=1}^{n}|x_i(t)|^p\right)^{\frac{1}{p}}, \tag{3}$$

where $p>1$ is a constant.

Throughout this paper, we assume that the activation functions f_j, g_j $(j=1,2,\cdots,n)$ satisfy the following property.

(H1): There exits positive constants m_j, n_j $(j=1,2,\cdots,n.)$ such that

$$|f_j(x)-f_j(y)|\leq m_j|x-y|;\quad |g_j(x)-g_j(y)|\leq n_j|x-y|,\quad \forall x,y\in\mathbf{R}. \tag{4}$$

(H2): $0<\tau_{ij}(t)<\tau$ and $\tau_{ij}^{'}(t)\leq\delta_{ij}<1$, $(i,j=1,2,\cdots,n.)$.

From (1), we can easily see that the solutions $x(t,\varphi)$ and $x(t,\psi)$ of system (1) through $(0,\varphi)$ and $(0,\psi)$ respectively satisfies the following equation

$$\begin{aligned}\tfrac{d}{dt}(x_i(t,\varphi)-x_i(t,\psi)) &= -a_i(t)(x_i(t,\varphi)-x_i(t,\psi))\\ &\quad+\sum_{j=1}^{n}w_{ij}(t)[f_j(x_j(t,\varphi))-f_j(x_j(t,\psi))]\\ &\quad+\sum_{j=1}^{n}w_{ij}^{\tau}(t)[g_j(x_j(t-\tau_{ij}(t),\varphi))-g_j(x_j(t-\tau_{ij}(t),\psi))],\end{aligned} \tag{5}$$

For the sake of convenience, we define

$$w_{ij}^{*}=\max\{|\underline{w}_{ij}|,|\overline{w}_{ij}|\},\ \ w_{ij}^{\tau*}=\max\{|\underline{w}_{ij}^{\tau}|,|\overline{w}_{ij}^{\tau}|\}.$$

To prove our main results, we need to introduce the following definition and lemma.

Definition 1. The periodic solution $x^*(t,\psi)$ of system (1) is said to be GRES, if for any initial state $\varphi\in C$ and for any $A(t)\in A_+,W(t)\in W_+,W^\tau(t)\in W_+^\tau$, there exist two positive constant $\lambda > 0, k \geq 1$, the state solution $x(t, \varphi)$ through $(0, \varphi)$ satisfies

$$\|x(t,\varphi) - x^*(t,\psi)\| \leq k\|\varphi - \psi\|e^{-\lambda t}, \ \forall t \geq 0. \tag{6}$$

Lemma 1.(Young's inequality) Assume that $a > 0, b > 0, p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality:

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q \tag{7}$$

holds.

3. Main results and Proofs

In this section, we shall show our main results and proofs.

Theorem 1. Assume that the hypothesis (H1) and (H2) hold, and suppose further that

(H3): there exists constants $q_{ij},\ q^*_{ij},\ r_{ij},\ r^*_{ij} \in \mathbf{R}$ $(i, j = 1, 2, \cdots, n.)$ and $\lambda_j > 0$ $(j = 1, 2, \cdots, n.)$ such that

$$\begin{aligned}&\sum_{j=1}^{n}\left[(p-1){w^{\tau}_{ij}}^{*\frac{p-q^*_{ij}}{p-1}} n_j^{\frac{p-r^*_{ij}}{p-1}} \frac{\lambda_j}{\lambda_i} {w^*_{ji}}^{q_{ji}} m_i^{r_{ji}}\right] \\ &+\sum_{j=1}^{n}\left[(p-1){w^*_{ij}}^{\frac{p-q_{ij}}{p-1}} m_j^{\frac{p-r_{ij}}{p-1}} + \frac{1}{1-\delta_{ji}}\frac{\lambda_j}{\lambda_i} {w^{\tau}_{ji}}^{*q^*_{ji}} n_i^{r^*_{ji}}\right] < p\underline{a}_i,\end{aligned}$$

for all $i = 1, 2, \cdots, n$. then the system (1) has a unique periodic solution $x^*(t, \psi)$ which is global robust exponential stable, where m_j, n_j $(j = 1, 2, \cdots, n.)$ are positive constants in (H1),

$$w^*_{ij} = \max\{|\underline{w}_{ij}|, |\overline{w}_{ij}|\},\ w^{\tau*}_{ij} = \max\{|\underline{w}^{\tau}_{ij}|, |\overline{w}^{\tau}_{ij}|\}, \quad i, j = 1, 2, \cdots, n.$$

Proof. Since

$$\begin{aligned}&-p\underline{a}_i + \sum_{j=1}^{n}\left[(p-1){w^{\tau}_{ij}}^{*\frac{p-q^*_{ij}}{p-1}} n_j^{\frac{p-r^*_{ij}}{p-1}} + \frac{\lambda_j}{\lambda_i} {w^*_{ji}}^{q_{ji}} m_i^{r_{ji}}\right] \\ &+\sum_{j=1}^{n}\left[(p-1){w^*_{ij}}^{\frac{p-q_{ij}}{p-1}} m_j^{\frac{p-r_{ij}}{p-1}} + \frac{\lambda_j}{\lambda_i}\frac{1}{1-\delta_{ji}} {w^{\tau}_{ji}}^{*q^*_{ji}} n_i^{r^*_{ji}}\right] < 0,\end{aligned}$$

for all $i = 1, 2, \cdots, n$. We can choose a small $\varepsilon > 0$ such that

$$\begin{aligned}&(\varepsilon - p\underline{a}_i) + \sum_{j=1}^{n}\left[(p-1){w^{\tau}_{ij}}^{*\frac{p-q^*_{ij}}{p-1}} n_j^{\frac{p-r^*_{ij}}{p-1}} + \frac{\lambda_j}{\lambda_i} {w^*_{ji}}^{q_{ji}} m_i^{r_{ji}}\right] \\ &+\sum_{j=1}^{n}\left[(p-1){w^*_{ij}}^{\frac{p-q_{ij}}{p-1}} m_j^{\frac{p-r_{ij}}{p-1}} + e^{\varepsilon\tau}\frac{\lambda_j}{\lambda_i}\frac{1}{1-\delta_{ji}} {w^{\tau}_{ji}}^{*q^*_{ji}} n_i^{r^*_{ji}}\right] < 0\end{aligned}$$

for all $i = 1, 2, \cdots, n$.

Construct the Lyapunov functional

$$\begin{aligned} V(t) &= \sum_{i=1}^{n} \lambda_i e^{\varepsilon t} |x_i(t,\varphi) - x_i(t,\psi)|^p \\ &\quad + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\lambda_j}{1-\delta_{ji}} w_{ji}^{\tau\ *q_{ji}^*} n_i^{r_{ji}^*} \int_{t-\tau_{ji}(t)}^{t} |x_i(s,\varphi) - x_i(s,\psi)|^p e^{\varepsilon(s+\tau)} ds, \end{aligned}$$

Set

$$V_1(t) = \sum_{i=1}^{n} \lambda_i e^{\varepsilon t} |x_i(t,\varphi) - x_i(t,\psi)|^p,$$

$$V_2(t) = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\lambda_j}{1-\delta_{ji}} w_{ji}^{\tau\ *q_{ji}^*} n_i^{r_{ji}^*} \int_{t-\tau_{ji}(t)}^{t} |x_i(s,\varphi) - x_i(s,\psi)|^p e^{\varepsilon(s+\tau)} ds,$$

then

$$V(t)) = V_1(t) + V_2(t).$$

Calculating the upper right derivative D^+V of V along the solution of system (3), we have

$$\begin{aligned} & D^+V_1(t)|_{(3)} \\ \leq\ & \varepsilon e^{\varepsilon t} \sum_{i=1}^{n} \lambda_i |x_i(t,\varphi) - x_i(t,\psi)|^p + e^{\varepsilon t} \sum_{i=1}^{n} p\lambda_i |x_i(t,\varphi) - x_i(t,\psi)|^{p-1} \\ & \times D^+|x_i(t,\varphi) - x_i(t,\psi)|_{(3)} \\ \leq\ & \varepsilon e^{\varepsilon t} \sum_{i=1}^{n} \lambda_i |x_i(t,\varphi) - x_i(t,\psi)|^p + e^{\varepsilon t} \sum_{i=1}^{n} p\lambda_i |x_i(t,\varphi) - x_i(t,\psi)|^{p-1} \\ & \times \mathrm{sign}[x_i(t,\varphi) - x_i(t,\psi)] \tfrac{d}{dt}[x_i(t,\varphi) - x_i(t,\psi)] \\ \leq\ & \varepsilon e^{\varepsilon t} \sum_{i=1}^{n} \lambda_i |x_i(t,\varphi) - x_i(t,\psi)|^p + e^{\varepsilon t} \sum_{i=1}^{n} p\lambda_i |x_i(t,\varphi) - x_i(t,\psi)|^{p-1} \\ & \times \mathrm{sign}[x_i(t,\varphi) - x_i(t,\psi)]\{-a_i(t)[x_i(t,\varphi) - x_i(t,\psi)] \\ & + \sum_{j=1}^{n} w_{ij}(t)[f_j(x_j(t,\varphi)) - f_j(x_j(t,\psi))] \\ & + \sum_{j=1}^{n} w_{ij}^{\tau}(t)[g_j(x_j(t-\tau_{ij}(t),\varphi)) - g_j(x_j(t-\tau_{ij}(t),\psi))]\}. \\ \leq\ & e^{\varepsilon t} \sum_{i=1}^{n} \lambda_i(\varepsilon - p\underline{a}_i)|x_i(t,\varphi) - x_i(t,\psi)|^p + e^{\varepsilon t} \sum_{i=1}^{n} p\lambda_i |x_i(t,\varphi) - x_i(t,\psi)|^{p-1} \\ & \times \left[\sum_{j=1}^{n} w_{ij}^* m_j |x_j(t,\varphi) - x_j(t,\psi)| + \sum_{j=1}^{n} w_{ij}^{\tau\ *} n_j |x_j(t-\tau_{ij}(t),\varphi) - x_j(t-\tau_{ij}(t),\psi)| \right] \\ \leq\ & e^{\varepsilon t} \sum_{i=1}^{n} \lambda_i(\varepsilon - p\underline{a}_i)|x_i(t,\varphi) - x_i(t,\psi)|^p \\ & + e^{\varepsilon t} p \sum_{i=1}^{n} \left[\lambda_i \sum_{j=1}^{n} w_{ij}^* m_j |x_i(t,\varphi) - x_i(t,\psi)|^{p-1} |x_j(t,\varphi) - x_j(t,\psi)| \right] \end{aligned}$$

$$
\begin{aligned}
&+e^{\varepsilon t}p\sum_{i=1}^{n}\left[\lambda_i\sum_{j=1}^{n}w_{ij}^{\tau}{}^{*}n_j|x_i(t,\varphi)-x_i(t,\psi)|^{p-1}|x_j(t-\tau_{ij}(t),\varphi)-x_j(t-\tau_{ij}(t),\psi)|\right]\\
=\;& e^{\varepsilon t}\sum_{i=1}^{n}\lambda_i(\varepsilon-p\underline{a}_i)|x_i(t,\varphi)-x_i(t,\psi)|^{p}\\
&+pe^{\varepsilon t}\sum_{i=1}^{n}\left[\lambda_i\sum_{j=1}^{n}w_{ij}^{*}{}^{\frac{p-q_{ij}}{p}}m_j^{\frac{p-r_{ij}}{p}}|x_i(t,\varphi)-x_i(t,\psi)|^{p-1}w_{ij}^{*}{}^{\frac{q_{ij}}{p}}m_j^{\frac{r_{ij}}{p}}|x_j(t,\varphi)-x_j(t,\psi)|\right]\\
&+pe^{\varepsilon t}\sum_{i=1}^{n}\Bigg[\lambda_i\sum_{j=1}^{n}w_{ij}^{\tau}{}^{*}{}^{\frac{p-q_{ij}^{*}}{p}}n_j^{\frac{p-r_{ij}^{*}}{p}}|x_i(t,\varphi)-x_i(t,\psi)|^{p-1}\\
&\times w_{ij}^{\tau}{}^{*}{}^{\frac{q_{ij}^{*}}{p}}n_j^{\frac{r_{ij}^{*}}{p}}|x_j(t-\tau_{ij}(t),\varphi)-x_j(t-\tau_{ij}(t),\psi)|\Bigg].
\end{aligned}
$$

By Lemma 1, we have

$$
\begin{aligned}
D^{+}V_1(t)|_{(3)}\;\le\;& e^{\varepsilon t}\sum_{i=1}^{n}\lambda_i(\varepsilon-p\underline{a}_i)|x_i(t,\varphi)-x_i(t,\psi)|^{p}\\
&+pe^{\varepsilon t}\sum_{i=1}^{n}\Bigg\{\lambda_i\sum_{j=1}^{n}\Bigg[\frac{1}{p}w_{ij}^{*}{}^{q_{ij}}m_j^{r_{ij}}|x_j(t,\varphi)-x_j(t,\psi)|^{p}\\
&+\frac{p-1}{p}w_{ij}^{*}{}^{\frac{p-q_{ij}}{p-1}}m_j^{\frac{p-r_{ij}}{p-1}}|x_i(t,\varphi)-x_i(t,\psi)|^{p}\Bigg]\Bigg\}\\
&+pe^{\varepsilon t}\sum_{i=1}^{n}\Bigg\{\lambda_i\sum_{j=1}^{n}\Bigg[\frac{1}{p}w_{ij}^{\tau}{}^{*}{}^{q_{ij}^{*}}n_j^{r_{ij}^{*}}|x_j(t-\tau_{ij}(t),\varphi)-x_j(t-\tau_{ij}(t),\psi)|^{p}\\
&+\frac{p-1}{p}w_{ij}^{\tau}{}^{*}{}^{\frac{p-q_{ij}^{*}}{p-1}}n_j^{\frac{p-r_{ij}^{*}}{p-1}}|x_i(t,\varphi)-x_i(t,\psi)|^{p}\Bigg]\Bigg\}\\
=\;& e^{\varepsilon t}\sum_{i=1}^{n}\lambda_i(\varepsilon-p\underline{a}_i)|x_i(t,\varphi)-x_i(t,\psi)|^{p}\\
&+e^{\varepsilon t}\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_j w_{ji}^{*}{}^{q_{ji}}m_i^{r_{ji}}|x_i(t,\varphi)-x_i(t,\psi)|^{p}\\
&+e^{\varepsilon t}\sum_{i=1}^{n}\sum_{j=1}^{n}(p-1)\lambda_i w_{ij}^{*}{}^{\frac{p-q_{ij}}{p-1}}m_j^{\frac{p-r_{ij}}{p-1}}|x_i(t,\varphi)-x_i(t,\psi)|^{p}\\
&+e^{\varepsilon t}\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_j w_{ji}^{\tau}{}^{*}{}^{q_{ji}^{*}}n_i^{r_{ji}^{*}}|x_i(t-\tau_{ji}(t),\varphi)-x_i(t-\tau_{ji}(t),\psi)|^{p}\\
&+e^{\varepsilon t}\sum_{i=1}^{n}\sum_{j=1}^{n}(p-1)\lambda_i w_{ij}^{\tau}{}^{*}{}^{\frac{p-q_{ij}^{*}}{p-1}}n_j^{\frac{p-r_{ij}^{*}}{p-1}}|x_i(t,\varphi)-x_i(t,\psi)|^{p}\\
=\;& e^{\varepsilon t}\sum_{i=1}^{n}\left\{(\varepsilon-p\underline{a}_i)\lambda_i+\sum_{j=1}^{n}\left[(p-1)\lambda_i w_{ij}^{\tau}{}^{*}{}^{\frac{p-q_{ij}^{*}}{p-1}}n_j^{\frac{p-r_{ij}^{*}}{p-1}}+\lambda_j w_{ji}^{*}{}^{q_{ji}}m_i^{r_{ji}}\right]\right\}\\
&\times|x_i(t,\varphi)-x_i(t,\psi)|^{p}\\
&+e^{\varepsilon t}\sum_{i=1}^{n}\sum_{j=1}^{n}(p-1)\lambda_i w_{ij}^{*}{}^{\frac{p-q_{ij}}{p-1}}m_j^{\frac{p-r_{ij}}{p-1}}|x_i(t,\varphi)-x_i(t,\psi)|^{p}\\
&+e^{\varepsilon t}\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_j w_{ji}^{\tau}{}^{*}{}^{q_{ji}^{*}}n_i^{r_{ji}^{*}}|x_i(t-\tau_{ji}(t),\varphi)-x_i(t-\tau_{ji}(t),\psi)|^{p}. \qquad (8)
\end{aligned}
$$

Again, note that $\tau'_{ji}(t) \le \delta_{ji} < 1,\ \tau_{ji}(t) < \tau\ (i,j = 1,2,\cdots,n.)$, we have

$$
\begin{aligned}
D^+V_2(t)|_{(3)} &= e^{\varepsilon t}\sum_{i=1}^{n}\sum_{j=1}^{n} e^{\varepsilon\tau}\tfrac{\lambda_j}{1-\delta_{ji}} w_{ji}^{\tau}{}^{*q_{ji}^*} n_i^{r_{ji}^*}|x_i(t,\varphi)-x_i(t,\psi)|^p \\
&\quad -e^{\varepsilon t}\sum_{i=1}^{n}\sum_{j=1}^{n}\tfrac{\lambda_j}{1-\delta_{ji}} w_{ji}^{\tau}{}^{*q_{ji}^*} n_i^{r_{ji}^*} e^{\varepsilon(-\tau_{ji}(t)+\tau)}|x_i(t-\tau_{ji}(t),\varphi)-x_i(t-\tau_{ji}(t),\psi)|^p \\
&\quad \times(1-\tau'_{ji}(t)). \\
&= e^{\varepsilon t}\sum_{i=1}^{n}\sum_{j=1}^{n} e^{\varepsilon\tau}\tfrac{\lambda_j}{1-\delta_{ji}} w_{ji}^{\tau}{}^{*q_{ji}^*} n_i^{r_{ji}^*}|x_i(t,\varphi)-x_i(t,\psi)|^p \\
&\quad -e^{\varepsilon t}\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_j w_{ji}^{\tau}{}^{*q_{ji}^*} n_i^{r_{ji}^*}|x_i(t-\tau_{ji}(t),\varphi)-x_i(t-\tau_{ji}(t),\psi)|^p. \qquad (9)
\end{aligned}
$$

From (8) and (9), we get

$$
\begin{aligned}
D^+V(t)|_{(3)} &\le D^+V_1(t)|_{(3)} + D^+V_2(t)|_{(3)} \\
&\le e^{\varepsilon t}\sum_{i=1}^{n}\left\{(\varepsilon - p\underline{a}_i)\lambda_i + \sum_{j=1}^{n}\left[(p-1)\lambda_i w_{ij}^{\tau}{}^{*\frac{p-q_{ij}^*}{p-1}} n_j^{\frac{p-r_{ij}^*}{p-1}} + \lambda_j w_{ji}^{*}{}^{q_{ji}} m_i^{r_{ji}}\right]\right\} \\
&\quad \times|x_i(t,\varphi)-x_i(t,\psi)|^p \\
&\quad + e^{\varepsilon t}\sum_{i=1}^{n}\sum_{j=1}^{n}(p-1)\lambda_i w_{ij}^{*}{}^{\frac{p-q_{ij}}{p-1}} m_j^{\frac{p-r_{ij}}{p-1}}|x_i(t,\varphi)-x_i(t,\psi)|^p \\
&\quad + e^{\varepsilon t}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\lambda_j}{1-\delta_{ji}} e^{\varepsilon\tau} w_{ji}^{\tau}{}^{*q_{ji}^*} n_i^{r_{ji}^*}|x_i(t,\varphi)-x_i(t,\psi)|^p \\
&= \sum_{i=1}^{n}\lambda_i\left\{(\varepsilon - p\underline{a}_i) + \sum_{j=1}^{n}\left[(p-1) w_{ij}^{\tau}{}^{*\frac{p-q_{ij}^*}{p-1}} n_j^{\frac{p-r_{ij}^*}{p-1}} + \frac{\lambda_j}{\lambda_i} w_{ji}^{*}{}^{q_{ji}} m_i^{r_{ji}}\right]\right. \\
&\quad \left. + \sum_{j=1}^{n}\left[(p-1) w_{ij}^{*}{}^{\frac{p-q_{ij}}{p-1}} m_j^{\frac{p-r_{ij}}{p-1}} + e^{\varepsilon\tau}\frac{\lambda_j}{\lambda_i}\frac{1}{1-\delta_{ji}} w_{ji}^{\tau}{}^{*q_{ji}^*} n_i^{r_{ji}^*}\right]\right\} \\
&\quad \times|x_i(t,\varphi)-x_i(t,\psi)|^p \\
&< 0, \qquad (10)
\end{aligned}
$$

Therefore,

$$V(t) \le V(0), \quad t \ge 0.$$

Since

$$e^{\varepsilon t}\min_{1\le i\le n}\{\lambda_i\}\sum_{i=1}^{n}|x_i(t,\varphi)-x_i(t,\psi)|^p \le V(t), \quad t \ge 0.$$

$$
\begin{aligned}
V(0) &= \sum_{i=1}^{n}\lambda_i|x_i(0,\varphi)-x_i(0,\psi)|^p + \sum_{i=1}^{n}\sum_{j=1}^{n}\tfrac{\lambda_j}{1-\delta_{ji}} w_{ji}^{\tau}{}^{*q_{ji}^*} n_i^{r_{ji}^*} \\
&\times \int_{-\tau_{ji}(0)}^{0}|x_i(s,\varphi)-x_i(s,\psi)|^p e^{\varepsilon(s+\tau)}ds
\end{aligned}
$$

$$\begin{aligned}
&\leq \max_{1\leq i\leq n}\{\lambda_i\}\sum_{i=1}^{n}|x_i(0,\varphi)-x_i(0,\psi)|^p + e^{\varepsilon\tau}\sum_{i=1}^{n}\sum_{j=1}^{n}\tfrac{\lambda_j}{1-\delta_{ji}}w_{ji}^{\tau}{}^{*q_{ji}^*}n_i^{r_{ji}^*}\\
&\times \int_{-\tau}^{0}|x_i(s,\varphi)-x_i(s,\psi)|^p e^{\varepsilon s}ds\\
&\leq \left[\max_{1\leq i\leq n}\{\lambda_i\}+\tau e^{\varepsilon\tau}\sum_{i=1}^{n}\sum_{j=1}^{n}\tfrac{\lambda_j}{1-\delta_{ji}}w_{ji}^{\tau}{}^{*q_{ji}^*}n_i^{r_{ji}^*}\right]\|\varphi-\psi)\|^p
\end{aligned}$$

Therefore,

$$\sum_{i=1}^{n}|x_i(t,\varphi)-x_i(t,\psi)|^p \leq e^{-\varepsilon t}\frac{\max\limits_{1\leq i\leq n}\{\lambda_i\}+\tau e^{\varepsilon\tau}\sum_{i=1}^{n}\sum_{j=1}^{n}\tfrac{\lambda_j}{1-\delta_{ji}}w_{ji}^{\tau}{}^{*q_{ji}^*}n_i^{r_{ji}^*}}{\min\limits_{1\leq i\leq n}\{\lambda_i\}}\|\varphi-\psi)\|^p$$

This implies that

$$\|x(t,\varphi)-x(t,\psi)\| \leq ke^{-\frac{\varepsilon}{p}t}\|\varphi-\psi)\| \tag{11}$$

where

$$k=\left\{\frac{\max\limits_{1\leq i\leq n}\{\lambda_i\}+\tau e^{\varepsilon\tau}\sum_{i=1}^{n}\sum_{j=1}^{n}\tfrac{\lambda_j}{1-\delta_{ji}}w_{ji}^{\tau}{}^{*q_{ji}^*}n_i^{r_{ji}^*}}{\min\limits_{1\leq i\leq n}\{\lambda_i\}}\right\}^{\frac{1}{p}}>1.$$

We can easily obtain from the formula (11) that

$$\|x_t(\varphi)-x_t(\psi)\| \leq ke^{-\frac{\varepsilon}{p}(t-\tau)}\|\varphi-\psi\|. \tag{12}$$

Now ,we can choose a positive integer m such that

$$ke^{-\frac{\varepsilon}{p}(m\omega-\tau)} \leq \frac{1}{2}. \tag{13}$$

Define a Poincar$\acute{e}$ mapping

$$P: C([-\tau,0],\mathbf{R}^n)\to C([-\tau,0],\mathbf{R}^n)$$

by $P\varphi=x_\omega(\varphi)$, then we can derive from (12) and (13) that

$$\|P^m\varphi-P^m\psi\|\leq\frac{1}{2}\|\varphi-\psi\|.$$

So, P^m is a contraction mapping, Thus, there exists a unique fixed point $x^*\in C([-\tau,0],\mathbf{R}^n)$ such that

$$P^m x^*=x^*.$$

Note that

$$P^m(Px^*)=P(P^m x^*)=Px^*.$$

It implies that $Px^*\in C([-\tau,0],\mathbf{R}^n)$ is also a fixed point of P^m. So,

$$Px^*=x^*, \quad \text{i.e.} \quad x_\omega(x^*)=x^*.$$

Let $x(t, x^*)$ be the solution of system (1) through $(0, x^*)$, obviously, $x(t+\omega, x^*)$ is also a solution of system (1) and note that

$$x_{t+\omega}(x^*) = x_t(x_\omega(x^*)) = x_t(x^*), \quad \text{for all } t \geq 0.$$

So,

$$x(t+\omega, x^*) = x(t, x^*), \quad \forall t \geq 0.$$

This shows that $x(t, x^*)$ is exactly one $\omega-$periodic solution of system (1) and it is easy to see from (12) that all solutions of system (1) converge globally exponentially to it as $t \to +\infty$. The proof is completed.

Corollary 1. Assume that the hypothesis (H1) and (H2) hold, $a_i(t) = a_i$, $w_{ij}(t) = w_{ij}$, $w_{ij}^\tau(t) = w_{ij}^\tau$, $I_i(t) = I_i$ $(i, j = 1, 2, \cdots, n.)$ are all time-invariant, and suppose further that

(H4): there exists constants $q_{ij},\ q_{ij}^*,\ r_{ij},\ r_{ij}^* \in \mathbf{R}$ $(i, j = 1, 2, \cdots, n.)$ and $\lambda_j > 0$ $(j = 1, 2, \cdots, n.)$ such that

$$\begin{aligned} &\sum_{j=1}^{n}\left[(p-1){w_{ij}^{\tau}}^{*\frac{p-q_{ij}^*}{p-1}} n_j^{\frac{p-r_{ij}^*}{p-1}} + \frac{\lambda_j}{\lambda_i}{w_{ji}^*}^{q_{ji}} m_i^{r_{ji}}\right] \\ &+\sum_{j=1}^{n}\left[(p-1){w_{ij}^*}^{\frac{p-q_{ij}}{p-1}} m_j^{\frac{p-r_{ij}}{p-1}} + \frac{1}{1-\delta_{ji}}\frac{\lambda_j}{\lambda_i}{w_{ji}^{\tau}}^{*q_{ji}^*} n_i^{r_{ji}^*}\right] < p\underline{a}_i, \end{aligned}$$

for all $i = 1, 2, \cdots, n.$ then the system (1) has a unique equilibrium point x^* which is global robust exponential stable, where m_j, n_j $(j = 1, 2, \cdots, n.)$ are positive constants in (H1).

Remark 1. Liao [4] study the system (1) in case that the activation function $f_j = g_j$ $(j = 1, 2, \cdots, n.)$ is monotone increasing and bounded and satisfying Lipschitz condition and $\tau_{ij}(t) = \tau_{ij} =$ constant. obtain the following result:

Theorem A: If there exist positive numbers $\lambda_1, \lambda_2, \cdots \lambda_n$ and $r_1 \in [0, 1]$, $r_2 \in [0, 1]$ such that

$$\alpha = \max_{1\leq i\leq n}\left\{\frac{1}{\underline{a}_i\lambda_i}\sum_{j=1}^{n}\left(\lambda_i L_j^{2r_1} w_{ij}^* + \lambda_j L_i^{2(1-r_1)} w_{ji}^* + \lambda_i L_j^{2r_2} {w_{ij}^{\tau}}^* + \lambda_j L_i^{2(1-r_2)} {w_{ji}^{\tau}}^*\right)\right\} < 2,$$

where

$$w_{ij}^* = \max\{|\underline{w}_{ij}|, |\overline{w}_{ij}|\},\ w_{ij}^{\tau*} = \max\{|\underline{w}_{ij}^{\tau}|, |\overline{w}_{ij}^{\tau}|\}, \quad i, j = 1, 2, \cdots, n.$$

then (1) with (2) has a unique globally robust stable equilibrium x^* for each constant input $I = (I_1, I_2, \cdots, I_n)^T \in \mathbf{R}^n$.

Clearly, this result is a special case of corollary 1. In fact, in corollary 1, we take $q_{ij}^* = q_{ij} = 1$, $r_{ij} = 2 - 2r_1$, $r_{ij}^* = 2 - 2r_2$, again take $f_j = g_j$, for all $j = 1, 2, \cdots, n$, then $m_j = n_j = L_j$, we can obtain the theorem A above.

Note that Theorem A need the activation functions to be monotone and bounded, our results do not request these assumptions. Therefore, our model generalizes those in [4].Moreover, our results are shown to be more effective then theorem A (see the example in next section).

Remark 2. Corollary 1 contain many previous results such as [5-8].

4. An Example

Consider the following system:

$$\begin{aligned} x_1'(t) &= -a_1x_1(t) + w_{11}f_1(x_1(t)) + w_{12}f_2(x_2(t)) + w_{11}^{\tau}g_1(x_1(t-\tau_1)) + w_{12}^{\tau}g_2(x_2(t-\tau_2)) + 1, \\ x_2'(t) &= -a_2x_1(t) + w_{21}f_1(x_1(t)) + w_{22}f_2(x_2(t)) + w_{21}^{\tau}g_1(x_1(t-\tau_1)) + w_{22}^{\tau}g_2(x_2(t-\tau_2)) + 2. \end{aligned}$$

We take $f_j = g_j = 0.5(|x+1| + |x-1|)$, $\tau_{ij} = \tau_j = \text{const.}$, $(i,j = 1,2.)$, then $m_j = n_j = 1$, $\delta_{ji} = 0$, $(i,j = 1,2.)$. Set

$$\underline{A} = \begin{pmatrix} \underline{a}_1 & 0 \\ 0 & \underline{a}_2 \end{pmatrix} = \begin{pmatrix} 0.95 & 0 \\ & 0 & 2 \end{pmatrix}, \quad \bar{A} = \begin{pmatrix} \bar{a_1} & 0 \\ 0 & \bar{a_2} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix},$$

$$\underline{W} = \begin{pmatrix} \underline{w}_{11} & \underline{w}_{12} \\ \underline{w}_{21} & \underline{w}_{22} \end{pmatrix} = \begin{pmatrix} 0.15 & -0.1 \\ 0.2 & 0.05 \end{pmatrix}, \quad \underline{W}^{\tau} = \begin{pmatrix} \underline{w}_{11}^{\tau} & \underline{w}_{12}^{\tau} \\ \underline{w}_{21}^{\tau} & \underline{w}_{22}^{\tau} \end{pmatrix} = \begin{pmatrix} 0.2 & 0.05 \\ 0.1 & 0.3 \end{pmatrix},$$

$$\bar{W} = \begin{pmatrix} \bar{w}_{11} & \bar{w}_{12} \\ \bar{w}_{21} & \bar{w}_{22} \end{pmatrix} = \begin{pmatrix} 0.2 & 0.3 \\ 0.4 & 0.1 \end{pmatrix}, \quad \bar{W}^{\tau} = \begin{pmatrix} \bar{w}_{11}^{\tau} & \bar{w}_{12}^{\tau} \\ \bar{w}_{21}^{\tau} & \bar{w}_{22}^{\tau} \end{pmatrix} = \begin{pmatrix} 0.3 & 0.1 \\ 0.2 & 0.1 \end{pmatrix},$$

$$W^* = \begin{pmatrix} w_{11}^* & w_{12}^* \\ w_{21}^* & w_{22}^* \end{pmatrix} = \begin{pmatrix} 0.2 & 0.3 \\ 0.4 & 0.1 \end{pmatrix}, \quad W^{\tau *} = \begin{pmatrix} {w_{11}^{\tau}}^* & {w_{12}^{\tau}}^* \\ {w_{21}^{\tau}}^* & {w_{22}^{\tau}}^* \end{pmatrix} = \begin{pmatrix} 0.3 & 0.1 \\ 0.2 & 0.3 \end{pmatrix},$$

where

$$w_{ij}^* = \max\{|\underline{w}_{ij}|, |\overline{w}_{ij}|\}, \; w_{ij}^{\tau *} = \max\{|\underline{w}_{ij}^{\tau}|, |\overline{w}_{ij}^{\tau}|\}. \quad (i,j = 1,2.)$$

In theorem 1, we take $p = 2$, choose $\lambda_i = 1$ $(i = 1,2.)$, $q_{11} = q_{12} = 1$, $q_{21} = q_{22} = 2$, and $q_{11}^* = q_{12}^* = q_{21}^* = q_{22}^* = 1$, then we obtain

$$\begin{aligned} & \sum_{j=1}^{2}\left[{w_{1j}^{\tau}}^{*2-q_{1j}^*} n_j^{2-r_{1j}^*} + \frac{\lambda_j}{\lambda_1}{w_{j1}^*}^{q_{j1}} m_1^{r_{j1}}\right] + \sum_{j=1}^{2}\left[{w_{1j}^*}^{2-q_{1j}} m_j^{2-r_{1j}} + \frac{1}{1-\delta_{j1}}\frac{\lambda_j}{\lambda_1}{w_{j1}^{\tau}}^{*q_{j1}^*} n_1^{r_{j1}^*}\right] \\ = \; & \sum_{j=1}^{2}\left[{w_{1j}^{\tau}}^{*2-q_{1j}^*} + {w_{j1}^*}^{q_{j1}}\right] + \sum_{j=1}^{2}\left[{w_{1j}^*}^{2-q_{1j}} + {w_{j1}^{\tau}}^{*q_{j1}^*}\right] = 1.66 < 1.9 = 2\underline{a}_1, \end{aligned}$$

and

$$\sum_{j=1}^{2}\left[{w_{2j}^{\tau}}^{*2-q_{2j}^{*}}n_j^{2-r_{2j}^{*}}+\frac{\lambda_j}{\lambda_2}{w_{j2}^{*}}^{q_{j2}}m_2^{r_{j2}}\right]+\sum_{j=1}^{2}\left[{w_{2j}^{*}}^{2-q_{2j}}m_j^{2-r_{2j}}+\frac{1}{1-\delta_{j2}}\frac{\lambda_j}{\lambda_2}{w_{j2}^{\tau}}^{*q_{j2}^{*}}n_2^{r_{j2}^{*}}\right]$$
$$=\sum_{j=1}^{2}\left[{w_{2j}^{\tau}}^{*2-q_{2j}^{*}}+{w_{j2}^{*}}^{q_{j2}}\right]+\sum_{j=1}^{2}\left[{w_{2j}^{*}}^{2-q_{2j}}+{w_{j2}^{\tau}}^{*q_{j2}^{*}}\right]=3.04<4=2\underline{a}_2.$$

Clearly, the conditions (H1)-(H3) hold. However, corresponding to theorem A, e.g., theorem 1 in [4]).

$$\begin{aligned}\alpha &= \max_{1\le i\le 2}\left\{\frac{1}{\underline{a}_i\lambda_i}\sum_{j=1}^{2}\left(\lambda_i L_j^{2r_1}w_{ij}^{*}+\lambda_j L_i^{2(1-r_1)}w_{ji}^{*}+\lambda_i L_j^{2r_2}{w_{ij}^{\tau}}^{*}+\lambda_j L_i^{2(1-r_2)}{w_{ji}^{\tau}}^{*}\right)\right\}\\ &= \max_{1\le i\le 2}\left\{\frac{1}{\underline{a}_i}\sum_{j=1}^{2}\left(w_{ij}^{*}+w_{ji}^{*}+{w_{ij}^{\tau}}^{*}+{w_{ji}^{\tau}}^{*}\right)\right\}\\ &= \max\{\frac{2}{0.95},0.5\}>2.\end{aligned}$$

So, the condition of theorem A does not hold. This example shows that our results improve and extend those in [4].

5. Conclusion

In this paper, we have given a family of sufficient conditions for global robust asymptotic stability and global robust exponential stability by applying Young inequality and general Lyapunov functional, and the conditions possess highly important significance in some applied fields, for instance, they can be applied to design globally exponentially stable CNNs with time-varying delays and easily checked in practice by simple algebraic methods. These play an important role in the design and applications of DCNNs. In addition, the methods in this paper may be applied to some other systems such as the systems given in Refs.[9-14] and so on.

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