Modeling Dynamical Systems via the Takagi-Sugeno Fuzzy Model

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Abstract: - The Takagi-Sugeno Fuzzy Model (TSF) is a universal approximator of the continuous real functions that are defined in a closed and bounded subset of R^n. This strong property of the (TSF) can find several applications modeling dynamical systems that can be described by differential equations. In the present paper, we apply TFS on dynamical systems (problems from Mathematical Ecology) and compare with then usual ODE's.

Key-Words: - Takagi-Sugeno model, Ordinary Differential Equations, Dynamical Systems, Mathematical Ecology

1 Introduction

The Takagi-Sugeno Fuzzy Model (TSF) is a universal approximator of the continuous real functions that are defined in a closed and bounded subset of R^n. That means that for each $\varepsilon > 0$ and for each continuous function g there exists a TFS such that $|g(x) - y(x)| < \varepsilon$ where $y(x)$ is the overall output of the TFS.

Some definitions and notation will be useful in the sequel. More details can be found in [1].

We denote the arbitrary Dynamical System as S and we can define some operations in the Systems' space.

Equality: The systems S_1 and S_2 are equal if and only if for the same input and the same initial conditions, the output of S_1 is equal to the output and S_2 ($\forall t \in (0,+\infty)$).

We write $S_1 = S_2$

Addition: The system S is the sum of S_1 and S_2, if and only if for the same input and the same initial conditions, the output of S is the sum of the outputs of S_1 and S_2 ($\forall t \in (0,+\infty)$).

Multiplication by a real constant: The system S is the product of the real number r and S_1, if and only if for the same input and the same initial conditions, the output of S is the product of the real number r and the output of S_1 ($\forall t \in (0,+\infty)$).

Product: The system S is the product of S_1 and S_2, if and only if for the same input and the same initial conditions, the output of S is the product of the outputs of S_1 and S_2 ($\forall t \in (0,+\infty)$).

Before proceeding in our analysis, we need also the definition of the term: Logical Expression (LE) Logical Expression or Logical Sentence is any expression or relation that is characterized true with degree μ of validity where $\mu \in [0,1]$ (μ=0 means absolutely false and μ=1 means absolutely true).

After these preliminary results and discussion, the simple or classical or common Takagi-Sugeno (TS) model is described as follows:

Suppose that we have the following n separate rules (that are independent each other)

If LE_i *then* S_i
where i=1,2,…,n

that means, for each rule (separately) if LE_i holds, then compute the entity y from the output of the system S_i with particular input and particular initial conditions for i=1,2,…,n.

In case, that we have simultaneously all the logical expressions LE_1, LE_2,…, LE_n with validity $\mu_1, \mu_2, ..., \mu_n$, then the entity y is computed from the output of the system S where:

$$S = \frac{\sum_{i=1}^{N} \mu_i S_i}{\sum_{i=1}^{N} \mu_i} \qquad (1)$$

In [1], the author has also presented several interesting concepts like Separable Additive Fuzzy Systems, Reciprocal Additive Fuzzy Systems, Separable Multiplicative Fuzzy Systems, Reciprocal Multiplicative Fuzzy Systems Differentiable Fuzzy Systems.

2 Modeling Dynamical Systems for Populations' Evolution using ODE's

Describing dynamics of population of one specie we usually use Ordinary Differential Equations. The following models are well known:

Malthus Model (Thomas Malthus, 1798):
We denote as x the population, i.e. the number of all individuals of one kind of organism (specie). Then the increase x' is proportional of x.

$$x' = ax \qquad (2)$$

where ax is the rate of increase (decrease, if a<0) of population. The ratio $\frac{x'}{x}$ is called growth rate and in Malthus Model is a constant a.

Model of Logistic growth (P. F. Verhust, 1838)
According to this model, when population size becomes high enough, there will be a density at which population growth stops altogether since real resources are limited. Consequently the growth rate cannot be a constant, it depends on the existing population and decrease with increasing value of population.

$$x' = a(x)x \qquad (3)$$

where $a(x)$ is a decreasing function of x. So, the growth rate is $a(x)$ and it is not constant but a function of x.

P.F.Verhurst proposed $a(x) = a - \beta x$ and so we have the model

$$x' = (a - \beta x)x \qquad (4)$$

Model of Populations with age structure (Hutchinson Model)

According to Hutchinson model the growth rate is considerably dependent on the age structure of the population. Therefore, a delay is introduced into Verhust equation to describe the age structure.

$$\frac{dx}{dt} = (a - \beta x(t - \tau))x(t) \qquad (5)$$

Harvesting Models
Harvesting models have played an important role in the "exploitation of renewable resources". An example is a fisherman. When fisherman has no way of locating the fish, fishing is a 'blind' way, the model takes the form

$$\frac{dx}{dt} = (a - \beta x(t))x(t) - Ex(t) \qquad (6)$$

where E is a proportionality constant, usually called the effort. The other way is 'purposeful', the fisherman knows precisely where the fish is and is able to decide how many fish he can catch, the model now takes the form

$$\frac{dx}{dt} = (a - \beta x(t))x(t) - E \qquad (7)$$

where E is the harvesting rate.

We can have also models of more than one species. Let's start with models of 2 species.
Lotka-Volterra Model (A.Lotka and V.Volterra 1926). According this model we have in a predator-prey community:

$$\frac{dx}{dt} = x(a - by) \qquad (8.1)$$

$$\frac{dy}{dt} = y(-c + dx) \qquad (8.2)$$

where y denotes the population of predator and x the population of prey. a, b, c, d are appropriate positive constants.

Model of Competitive Species
If instead if predator-prey relation, we have mutual competion, the model must be modified as follows

$$\frac{dx}{dt} = x(a - a_{11}x - a_{12}y) \qquad (9.1)$$

$$\frac{dy}{dt} = y(b - a_{21}x - a_{22}y) \qquad (9.2)$$

where x,y are the two populations and a, b, a_{ij} $i,j = 1,2$ are positive constants.

Generalizing the previous models, we can have the following general model of Kolmogorov.

Kolmogorov's model

$$\frac{dx}{dt} = xf(x, y) \qquad (10.1)$$

$$\frac{dy}{dt} = yg(x, y) \qquad (10.2)$$

where x,y are the two populations and f,g are appropriate differentiable (with respect x,y) functions. Some other models in [3] and [4] can be considered as special cases of the Kolmogorov's model. The sign of the partial derivatives of f,g determine the biological behaviour of the model and therefore the classification of the particular community.

3 Modeling Dynamical Systems using TFS. Comparison with ODE's

In this section, we use TSF for dynamical systems' description. As examples we consider populations' systems (ecological systems) and we compare the TSF modeling with ODE's.

Malthus Model
Suppose that we have a population x and the following n separate fuzzy rules (we can consider them as result of our statistics or measurements)

Rule 0: If $x \in [x_0, x_1]$, then $\frac{dx}{dt} = a_0$

Rule 1: If $x \in [x_0, x_2]$, then $\frac{dx}{dt} = a_1$

Rule 2: If $x \in [x_1, x_3]$, then $\frac{dx}{dt} = a_2$

Rule 3: If $x \in [x_2, x_4]$, then $\frac{dx}{dt} = a_3$

Rule i: If $x \in [x_{i-1}, x_{i+1}]$, then $\frac{dx}{dt} = a_i$

Rule n-1: If $x \in [x_{n-2}, x_n]$, then $\frac{dx}{dt} = a_{n-1}$

Rule n: If $x \in [x_{n-1}, x_n]$, then $\frac{dx}{dt} = a_n$

For the Rule 0, i.e. $x \in [x_0, x_1]$, the membership function (that is the degree of validity of the rule) is:

$$\mu_0(x)=\begin{cases}1-\dfrac{(x-x_0)}{x_1-x_0} & for\ x\in[x_0.x_1]\\ 0 & otherwise\end{cases}$$

For the Rule 1, the membership function (that is the degree of validity of the rule) is:

$$\mu_1(x)=\begin{cases}\dfrac{x-x_0}{a_1-a_0} & for\ x\in[x_0.x_1]\\ 1-\dfrac{x-x_1}{x_2-x_1} & for\ x\in[x_2.x_1]\\ 0 & otherwise\end{cases}$$

In general, for the Rule i (i=2,3,…n-1), the membership function is:

$$\mu_i(x)=\begin{cases}\dfrac{x-x_{i-1}}{x_i-x_{i-1}} & for\ x\in[x_{i-1}.x_i]\\ 1-\dfrac{x-x_i}{x_{i+1}-x_i} & for\ x\in[x_i.x_{i+1}]\\ 0 & otherwise\end{cases}$$

For the Rule n, the membership function is:

$$\mu_n(x)=\begin{cases}\dfrac{x-x_{n-1}}{x_n-x_{n-1}} & for\ x\in[x_{n-1}.x_n]\\ 0 & otherwise\end{cases}$$

So, according the TSF, the overall model is:

$$\frac{dx}{dt}=\frac{\mu_0 a_0+\mu_1 a_1+...+\mu_n a_n}{\mu_0+\mu_1+...+\mu_n}$$

Since $\mu_0,\mu_1,...,\mu_n$ vanishes in several intervals we can find that

$$\frac{dx}{dt}=(1-\frac{(x-x_0)}{x_1-x_0})\,a_0+\frac{x-x_0}{x_1-x_0}a_1 \quad for\ x\in[x_0.x_1]$$

or after some simple algebraic manipulation

$$\frac{dx}{dt}=(1-\frac{(x-x_0)}{x_1-x_0})\,a_0+\frac{x-x_0}{x_1-x_0}a_1 \quad for\ x\in[x_0.x_1]$$

Quite similarly

$$\frac{dx}{dt}=a_1+\frac{x-x_1}{x_2-x_1}(a_2-a_1) \quad for\ x\in[x_1.x_2]$$

and in general we can obtain:

$$\frac{dx}{dt}=a_{i-1}+\frac{a_i-a_{i-1}}{x_i-x_{i-1}}(x-x_{i-1}) \quad for\ x\in[x_{i-1},x_i] \tag{11}$$

for i=1,2,…,n

Equation (2), i.e. the Malthus model can be considered now as a special case of (11) when $x_0=0$, $a_0=0$ and the $\frac{a_i-a_{i-1}}{x_i-x_{i-1}}=constant$ for all i (suppose a). Then, the right-hand side of (11) can be simplified to ax because of the continuity of the straight lines of $a_{i-1}+\frac{a_i-a_{i-1}}{x_i-x_{i-1}}(x-x_{i-1})$ and the equal coefficients $\frac{a_i-a_{i-1}}{x_i-x_{i-1}}$.

The conclusion is that the Takagi-Sugeno model gives a more realistic and more reliable model as well as the Malthus model (2) can be considered now as a special case.

We consider now the so-called growth rate: $\tilde{x}=\frac{\frac{dx}{dt}}{x}$ and suppose that our statistics (measurements) yields the following rules:

Rule 0: If $x\in[x_0,x_1]$, then $\tilde{x}=a_0$

Rule 1: If $x\in[x_0,x_2]$, then $\tilde{x}=a_1$

Rule i: If $x\in[x_{i-1},x_{i+1}]$, then $\tilde{x}=a_i$

Rule n-1: If $x \in [x_{n-2}, x_n]$, then $\tilde{x} = a_{n-1}$

Rule n: If $x \in [x_{n-1}, x_n]$, then $\tilde{x} = a_n$

(In order to have a real model, useful in mathematical ecology, we demand: $a_0 \neq 0$ and $a_0 > a_1 > ... > a_n$)

For the Rule i (i=2,3,…n-1), the membership function is:

$$\mu_i(x) = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}} & for\ x \in [x_{i-1}.x_i] \\ 1 - \frac{x - x_i}{x_{i+1} - x_i} & for\ x \in [x_i.x_{i+1}] \\ 0 & otherwise \end{cases}$$

while:

$$\mu_0(x) = \begin{cases} 1 - \frac{(x - x_0)}{x_1 - x_0} & for\ x \in [x_0.x_1] \\ 0 & otherwise \end{cases}$$

$$\mu_n(x) = \begin{cases} \frac{x - x_{n-1}}{x_n - x_{n-1}} & for\ x \in [x_{n-1}.x_n] \\ 0 & otherwise \end{cases}$$

So, according the TSF, the overall model is:

$$\tilde{x} = \frac{\frac{dx}{dt}}{x} = \frac{\mu_0 a_0 + \mu_1 a_1 + ... \mu_n a_n}{\mu_0 + \mu_1 + ... \mu_n}$$

Therefore:

$$\frac{dx}{dt} = \left(a_{i-1} + \frac{a_i - a_{i-1}}{x_i - x_{i-1}} (x - x_{i-1}) \right) x \quad for\ x \in [x_{i-1}, x_i] \tag{12}$$

for i=1,2,…,n

Equation (2), i.e. the Verhurst model, can be also considered as a special case of (11) when $x_0 = 0$, $a_0 \neq 0$ and $a_0 > a_1 > ... > a_n$ and the $\frac{a_{i-1} - a_i}{x_i - x_{i-1}}$ = *constant*=*a*. Then, the right-hand side of (11) can be simplified to $(a_0 - ax)x$ because of the continuity of the straight lines of $a_{i-1} + \frac{a_i - a_{i-1}}{x_i - x_{i-1}}(x - x_{i-1})$ and the equal coefficients $\frac{a_i - a_{i-1}}{x_i - x_{i-1}}$.

The conclusion is that the Takagi-Sugeno model gives a more realistic and more reliable model as well as the Verhurst model (4) can be considered now as a special case.

Working similarly we can formulate now similar results for model of populations with age structure.

$$\frac{dx(t)}{dt} = \left(a_{i-1} + \frac{a_i - a_{i-1}}{x_i - x_{i-1}} \left(x(t - \tau) - x_{i-1}(t - \tau) \right) \right) x(t) \quad for\ x \in [x_{i-1}, x_i] \tag{13}$$

for i=1,2,…,n

The conclusion is again that the Takagi-Sugeno model gives a more realistic and more reliable model as well as the age structure model (5) can be considered now as a special case.

For the Harvesting Models, suppose that our measurement (statistics) can give us a number a_i and a number E_i for each interval. We have for the first harvesting model, Eq.(6), :

$$\frac{dx}{dt} = \left(a_{i-1} + \frac{a_i - a_{i-1}}{x_i - x_{i-1}}(x - x_{i-1}) - E_i \right) x$$
$$for \ x \in [x_{i-1}, x_i] \qquad (14)$$

for i=1,2,…,n

where we considered again the growth rate ($\tilde{x} = \frac{\frac{dx}{dt}}{x}$) in the considered TSF model

For the second harvesting model, Eq.(7), we make first some modified consideration: From (7), we need to define as:

$$\hat{x} = \frac{\frac{dx(t)}{dt} + E_i}{x(t)} \qquad (15)$$

So, we can find:

$$\frac{dx}{dt} = \left(a_{i-1} + \frac{a_i - a_{i-1}}{x_i - x_{i-1}}(x - x_{i-1}) \right) x - E_i$$
$$for \ x \in [x_{i-1}, x_i] \qquad (16)$$

for i=1,2,…,n

In this case, we observe that the Takagi-Sugeno model can gives more realistic and reliable results. In the special case, where $\frac{a_{i-1} - a_i}{x_i - x_{i-1}}$ = *constant* we have the (simple) harvesting models.

We can now examine the general case of populations of 2 species, considering the general model of Kolmogorov (Eq.10.1 and 10.2). The growth rates are now considered

$$\tilde{x} = \frac{\frac{dx}{dt}}{x} = f(x, y) \quad \text{and}$$

$$\tilde{y} = \frac{\frac{dy}{dt}}{y} = g(x, y)$$

We can consider the Takagi-Sugeno model for *f*, *y* separately. Therefore

<u>Rule *i,j*:</u> If $(x, y) \in [x_{i-1}, x_{i+1}] \times [y_{j-1}, y_{j+1}]$, then $f = a_{i,j}$
(where 1<*i*<*n*, 1<*j*<*m*)

The membership function (i.e. the validity degree) of this rule is $\mu(x, y)$ and can be considered as the product of $\mu_{1i}(x)$ by $\mu_{2j}(y)$, where $\mu_{1i}(x)$ and $\mu_{2j}(y)$ are given by

$$\mu_{1i}(x) = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}} & for \ x \in [x_{i-1}, x_i] \\ 1 - \frac{x - x_i}{x_{i+1} - x_i} & for \ x \in [x_i, x_{i+1}] \\ 0 & otherwise \end{cases}$$

while:

$$\mu_{10}(x) = \begin{cases} 1 - \frac{(x - x_0)}{x_1 - x_0} & for \ x \in [x_0, x_1] \\ 0 & otherwise \end{cases}$$

$$\mu_{1n}(x) = \begin{cases} \frac{x - x_{n-1}}{x_n - x_{n-1}} & for \ x \in [x_{n-1}, x_n] \\ 0 & otherwise \end{cases}$$

$$\mu_{2j}(y)=\begin{cases}\dfrac{y-y_{j-1}}{y_j-y_{j-1}} & \textit{for } y\in[y_{j-1},y_j]\\ 1-\dfrac{y-y_j}{y_{j+1}-y_j} & \textit{for } x\in[y_j,y_{j+1}]\\ 0 & \textit{otherwise}\end{cases}$$

while:

$$\mu_{20}(y)=\begin{cases}1-\dfrac{(y-y_0)}{y_1-y_0} & \textit{for } y\in[y_0,y_1]\\ 0 & \textit{otherwise}\end{cases}$$

$$\mu_{2n}(y)=\begin{cases}\dfrac{y-y_{m-1}}{y_m-y_{m-1}} & \textit{for } y\in[y_{m-1},y_m]\\ 0 & \textit{otherwise}\end{cases}$$

Applying now the Takagi-Sugeno model, we obtain for $(x,y)\in[x_{i-1},x_i]\times[y_{j-1},y_j]$ the following:

$$f=\left(1-\frac{x-x_{i-1}}{x_i-x_{i-1}}\right)\left(1-\frac{y-y_{i-1}}{y_i-y_{i-1}}\right)a_{i-1,j-1}+$$
$$+\left(\frac{x-x_{i-1}}{x_i-x_{i-1}}\right)\left(1-\frac{y-y_{i-1}}{y_i-y_{i-1}}\right)a_{i,j-1}+$$
$$+\left(1-\frac{x-x_{i-1}}{x_i-x_{i-1}}\right)\left(\frac{y-y_{i-1}}{y_i-y_{i-1}}\right)a_{i-1,j}+$$
$$+\left(\frac{x-x_{i-1}}{x_i-x_{i-1}}\right)\left(\frac{y-y_{i-1}}{y_i-y_{i-1}}\right)a_{i,j}$$

So,

$$f=a_{i-1,j-1}+\left(\frac{x-x_{i-1}}{x_i-x_{i-1}}\right)(a_{i,j-1}-a_{i-1,j-1})+$$
$$+\left(\frac{y-y_{j-1}}{y_j-y_{j-1}}\right)(a_{i-1,j}-a_{i-1,j-1})+$$
$$+\left(\frac{x-x_{i-1}}{x_i-x_{i-1}}\right)\left(\frac{y-y_{j-1}}{y_j-y_{j-1}}\right)(a_{i,j}-a_{i-1,j}-a_{i,j-1}+a_{i-1,j-1})$$

(17)

where $a_{i,j}$ are the values of the values of *f* obtained on (*i,j*) point by our measurements (statistics).

Quite similar expression for *g* can be obtained

$$g=b_{i-1,j-1}+\left(\frac{x-x_{i-1}}{x_i-x_{i-1}}\right)(b_{i,j-1}-b_{i-1,j-1})+$$
$$+\left(\frac{y-y_{j-1}}{y_j-y_{j-1}}\right)(b_{i-1,j}-b_{i-1,j-1})+$$
$$+\left(\frac{x-x_{i-1}}{x_i-x_{i-1}}\right)\left(\frac{y-y_{j-1}}{y_j-y_{j-1}}\right)(b_{i,j}-b_{i-1,j}-a_{i,j-1}+a_{i-1,j-1})$$

(18)

where $b_{i,j}$ are the values of the values of *f* obtained on (*i,j*) point by our measurements (statistics).
Analogously the sign of $a_{i,j}$, $b_{i,j}$ and the sign and their dependence of the differences ($a_{i,j-1}-a_{i-1,j-1}$, $b_{i,j-1}-b_{i-1,j-1}$, …) on *x,y,* we can classified the various models resulting from this general model of (10.1) and (10.2) with *f, g* given by (17) and (18).

4 Conclusion

The conclusion is that the Takagi-Sugeno model can give in all cases a more realistic and reliable model for modelling of dynamical systems (as an example we modelled populations of ecosystems including the general case of more than one specie). On the other hand, usual dynamical models (for example the previously published models of mathematical ecology) can be considered now as special cases of the Takagi-Sugeno model.

References:

[1] N.E. Mastorakis "General Fuzzy Systems as extensions of the Takagi-Sugeno methodology", Proceedings of the 4th WSEAS *International Conference on Fyzzy Sets and Fuzzy Systems,* Udine, Italy, march 25-27, 2004.

[2] T. Takagi and M. Sugeno, "Fuzzy identification of systems and its applications to modeling and control," IEEE Trans. Syst., Man, Cybern., vol. 15, pp. 116–132, 1985.

[3] R. F. Luck, Evaluation of natural enemies for biological control: a behavior approach, Trends in Ecology and Evolution, 5, pp. 196-199, 1990

[4] M. L. Rosenzweig, Paradox of enrichment: destabilization of exploitation systems in ecological time, Science 171, pp. 385–387, 1969.

[5] D. P. Filev and R. R. Yager, "A generalized defuzzification method via BAD distributions," Int. J. Intell. Syst., vol. 6, pp. 687–697, 1991.

[6] R.R. Yager, D.P.Filev, Essential of Fuzzy Modeling and Control, Wiley, New York, 1994.

[7] Hao Ying, "General SISO Takagi–Sugeno Fuzzy Systems with Linear Rule Consequent are Universal Approximators", IEEE Transactions on Fuzzy Systems, vol. 6, no. 4, pp.582-587, November 1998

[8] Hao Ying, "Sufficient Conditions on Uniform Approximation of Multivariate Functions by General Takagi–Sugeno Fuzzy Systems with Linear Rule Consequent", IEE Trans. on Systems, Man, and Cybernetics—Part A: Systems and Humans, Vol. 28, No. 4, pp.515-520, July 1998

[9] Robert Babuska, Fuzzy Modeling for Control, Kluwer Academic Publishers, Norwell, MA, 1998 .

[10] D. Driankov , H. Hellendoorn , M. Reinfrank, An introduction to fuzzy control, Springer-Verlag New York, Inc., New York, NY, 1993

[11] George J. Klir , Bo Yuan, Fuzzy sets and fuzzy logic: theory and applications, Prentice-Hall, Inc., Upper Saddle River, NJ, 1994

[12] Bart Kosko, Fuzzy engineering, Prentice-Hall, Inc., Upper Saddle River, NJ, 1996

[13] M. Sugeno, Fuzzy Modeling and Control. Selected Works of M. Sugeno, Computing Engineering, CRC, 1999