General Fuzzy Systems as extensions of the Takagi-Sugeno methodology

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Abstract: - In this paper, we discuss a general fuzzy model that is based on the Takagi-Sugeno inference engine. In particular, we propose new general inference mechanisms that are not straightforward extensions of the singleton model inference (as Takagi-Sugeno first model). We present also several interesting concepts like: Separable Additive Fuzzy Systems, Reciprocal Additive Fuzzy Systems, Separable Multiplicative Fuzzy Systems, Reciprocal Multiplicative Fuzzy Systems Differentiable Fuzzy Systems. Some examples are given along with hints for future research.

Key-Words: - General Fuzzy Systems, Takagi-Sugeno model, Separable Additive Fuzzy Systems, Reciprocal Additive Fuzzy Systems, Separable Multiplicative Fuzzy Systems, Reciprocal Multiplicative Fuzzy Systems Differentiable Fuzzy Systems.

1 Introduction

The Takagi-Sugeno (TS) fuzzy model is a well-known method of fuzzy reasoning with numerous applications in fuzzy systems, fuzzy control and in general in fuzzy engineering. The main advantage of the TS model is that is a simple model and can be used in many practical applications for modeling and control of complex systems. Due to its importance, the TS model has attracted much attention among theoretical academic scientists as well as engineers that work in industry and applied research.

The simple or classical or common TS model includes linear rule consequent and the centroid defuzzifier as originally proposed by Takagi and Sugeno [1]. The interesting existence result in [2] is on an uncommon two-input one-output TS fuzzy system because it uses a linear defuzzifier, (i.e., the defuzzifier does not have denominator) and requires rule consequent be polynomials of the input variables. In [3] and [4] a first generalization of the simple TS model is proposed. In [5] and [6], H. Ying proves that the General TS Systems of [3] and [4] are universal approximators in the Stone-Weierstrass sense. That means that every continuous function defined on an interval [a,b] can be uniformly approximated as closely as desired by the output of a General TS system. (The Stone-Weierstrass approximation theorem states that every continuous function defined on an interval [a,b] can be uniformly approximated as closely as desired by a polynomial function).

In this paper, we attempt to make more extensions in the generalization of [3] and [4] and present the most general Fuzzy Model which is also universal approximators in the Stone-Weierstrass sense.

Some hints for practical applications and directions for future research are also given.

2 The simple Takagi-Sugeno Model

Before proceeding we must introduce some definitions and notation.
We denote the arbitrary Dynamical System as S and we can define some operations in the Systems' space.
Equality: The systems S_1 and S_2 are equal if and only if for the same input and the same initial conditions, the output of S_1 is equal to the output and S_2 ($\forall t \in (0,+\infty)$). We write $S_1 = S_2$
Addition: The system S is the sum of S_1 and S_2, if and only if for the same input and the

same initial conditions, the output of S is the sum of the outputs of S_1 and S_2 ($\forall t \in (0,+\infty)$).

Multiplication by a real constant: The system S is the product of the real number r and S_1, if and only if for the same input and the same initial conditions, the output of S is the product of the real number r and the output of S_1 ($\forall t \in (0,+\infty)$).

Product: The system S is the product of S_1 and S_2, if and only if for the same input and the same initial conditions, the output of S is the product of the outputs of S_1 and S_2 ($\forall t \in (0,+\infty)$).

Stable System: A System S is called stable if for every bounded input (bounded function of time) the output is also bounded (bounded function of time).

Let us denote by U the universal set of all general Dynamical Systems with bounded inputs (bounded by a particular positive constant M) and zero initial conditions.

In U, we can define a function of the system S as follows $\|S\| = \sup(\|y\|_n)$ for all different bounded inputs (functions of time t) and zero initial conditions. It is apparent that this function is a norm because

- $\|S\| > 0 \; if \; S \neq 0$ and $\|S\| = 0 \; if \; S = 0$
- $\|\lambda S\| = |\lambda| \|S\|$ for any real number λ
- $\|S_1 + S_2\| \leq \|S_1\| + \|S_2\|$ (triangle inequality)

So, U is a normed space. On the other hand, U is a also a vector space, as one can easily verify using the aforementioned definitions for the Addition and Multiplication by a real constant.

So, U is a metric vector space.

Let's consider now a sequence of systems $S_1, S_2, ...S_k$ with the special property:

$\forall \varepsilon > 0, \exists \nu$ such that $d_n(S_l, S_m) < \varepsilon$ for each $l, m \geq \nu$. We can prove that there exists a system S that it is the limit of this sequence. The systems is defined as the system the output of which (for the particular bounded input and for zero initial conditions) is the limit of the outputs (for the particular bounded input and for zero initial conditions). That proves that U is a complete normed vector space, i.e. U is a **Banach Space.**

Before proceeding in our analysis, we need the definition of the term: Logical Expression (LE)

Logical Expression or Logical Sentence is any expression or relation that is characterized true with degree μ of validity where $\mu \in [0,1]$ (μ=0 means absolutely false and μ=1 means absolutely true).

After these preliminary results and discussion, the simple or classical or common Takagi-Sugeno (TS) model is described as follows:

Suppose that we have the following n separate fuzzy rules (that are independent each other)

If LE_i *then* S_i

where i=1,2,…,n

that means, for each rule (separately) if LE_i holds, then compute the entity y from the output of the system S_i with particular input and particular initial conditions for i=1,2,…,n.

In case, that we have simultaneously all the logical expressions LE_1, LE_2,…, LE_n with validity $\mu_1, \mu_2, ..., \mu_n$, then the entity y is computed from the output of the system S where:

$$\boxed{S = \frac{\sum_{i=1}^{N} \mu_i S_i}{\sum_{i=1}^{N} \mu_i} \qquad (1)}$$

The literature of fuzzy systems and fuzzy control is full of applications of Takagi-Sugeno model (1).

2 The general Takagi-Sugeno Model

In [3] and [4], a first generalization of the simple TS model is extended in a more general TS model as follows:
Suppose that we have the following *n* separate and fuzzy rules (that are independent each other)

If LE_i *then* S_i
(where i=1,2,…,n)

that means, for each rule (separately) if LE_i holds, then compute the entity *y* from the output of the system S_i with particular input and particular initial conditions for i=1,2,…,n.

In case, that we have simultaneously all the logical expressions LE_1, LE_2,…, LE_n with validity $\mu_1, \mu_2, ..., \mu_n$, then the entity *y* is given from the output of the system *S* where:

$$S = \frac{\sum_{i=1}^{N} \mu_i^{\rho} S_i}{\sum_{i=1}^{N} \mu_i^{\rho}} \tag{2}$$

where ρ is a positive integer constant. For ρ=1, we have the simple TS model, while for $\rho \to \infty$, we take the very interesting case:

$$S = S_{\bar{\mu}} \tag{3}$$

where $\bar{\mu} = \max(\mu_1, \mu_2, ..., \mu_n)$.

It has been proved by Ying, in [5] and [6], that the General TS Systems of (2) are universal approximators in the Stone-Weierstrass sense (for each value of ρ). That means that every continuous function defined on an interval [a,b] can be uniformly approximated as closely as desired by the entity *y* of a general TS system.

3 Function of Systems and the (general) Fuzzy System

Function of Systems: We can define now a function $\mathbf{f}$ from U^n into U mapping the system $S_1, ..., S_n$ to the system $\mathbf{f}(S)$

$$(S_1, ..., S_n) \to \mathbf{f}(S_1, ..., S_n).$$

Let us consider now the mapping from $R^n \times \mathsf{U}^n$ into U, where R^n is the R^n space of $\mu_1, \mu_2, ..., \mu_n$ (the values of the validity of LE_1, LE_2,…, LE_n) and U^n is the cartesian product of *n* U's.

We can use also the same symbol $\mathbf{f}$

$$(\mu_1, \mu_2, ..., \mu_n, S_1, ..., S_n) \to \mathbf{f}(\mu_1, \mu_2, ..., \mu_n, S_1, ..., S_n).$$

As we can see, (1), (2), (3) are now various subcases of $\mathbf{f}$.

(General) Fuzzy System

The system $\mathbf{f}$ $(\mu_1, \mu_2, ..., \mu_n, S_1, ..., S_n)$ where $\mu_1, \mu_2, ..., \mu_n$ are the validity of LE_1, LE_2,…, LE_n logical sentences ($\mu_i \in [0,1]$) and $S_1, ..., S_n$ are arbitrary dynamical systems is called (general) fuzzy system.

4 Additive Fuzzy Systems. Convex Additive Fuzzy Systems

Additive Fuzzy System is the Fuzzy System $\mathbf{f}(\mu_1, \mu_2, ..., \mu_n, S_1, ..., S_n)$ where $\mu_1, \mu_2, ..., \mu_n$ are the values of the validity of LE_1, LE_2,…, LE_n logical sentences ($\mu_i \in [0,1]$) and $S_1, ..., S_n$ are arbitrary dynamical systems if and only if:

$$\mathbf{f}(\mu_1, \mu_2, ..., \mu_n, S_1, ..., S_n) = f_1(\mu_1, \mu_2, ..., \mu_n) S_1 + ... + f_n(\mu_1, \mu_2, ..., \mu_n) S_n$$

Convex Additive Fuzzy System is called an Additive Fuzzy System where we have additionally the equation:
$f_1(\mu_1, \mu_2,..., \mu_n) + ... + f_n(\mu_1, \mu_2,..., \mu_n) = 1$

Remark: The simple and the general Takagi-Sugeno System are convex fuzzy systems.

Remark: The concepts of the session 4 can also be found in [10].

5 Separable Additive Fuzzy Systems

Suppose the Additive Fuzzy System $\mathbf{f}(\mu_1, \mu_2,..., \mu_n, S_1,..., S_n)$ where $\mu_1, \mu_2,..., \mu_n$ are the validity of LE_1, LE_2,…, LE_n ($\mu_i \in [0,1]$) and $S_1,..., S_n$ are arbitrary dynamical systems.

The Additive Fuzzy System $\mathbf{f}$ is called Separable Additive Fuzzy System if and only if

- f_i is depended only on μ_i, $\forall i = 1,2,..., n$, but it does not depended on any μ_j $\forall i, j,\ i \neq j$

6 Reciprocal Additive Fuzzy Systems

Suppose the Additive Fuzzy System $\mathbf{f}(\mu_1, \mu_2,..., \mu_n, S_1,..., S_n)$ where $\mu_1, \mu_2,..., \mu_n$ are the validity of LE_1, LE_2,…, LE_n ($\mu_i \in [0,1]$) and $S_1,..., S_n$ are arbitrary dynamical systems.

The Additive Fuzzy System $\mathbf{f}$ is called Reciprocal Additive Fuzzy System if and only if

- f_i is dependented on μ_i in the same way that f_j is dependented on μ_j $\forall i, j$
- f_i remains invariant if we interchange the variables μ_k, μ_l $\forall i, k, l$ with $k \neq i, l \neq i$

7 Multiplicative Fuzzy Systems.

Multiplicative Fuzzy System is the Fuzzy System $\mathbf{f}(\mu_1, \mu_2,..., \mu_n, S_1,..., S_n)$ where $\mu_1, \mu_2,..., \mu_n$ are the values of the validity of LE_1, LE_2,…, LE_n ($\mu_i \in [0,1]$) and $S_1,..., S_n$ are arbitrary dynamical systems if and only if:

$\mathbf{f}(\mu_1, \mu_2,..., \mu_n, S_1,..., S_n)=$
$f_1(\mu_1, \mu_2,..., \mu_n)...f_n(\mu_1, \mu_2,..., \mu_n)S_1...S_n$

8 Separable Multiplicative Fuzzy Systems

Suppose the Multiplicative Fuzzy System $\mathbf{f}(\mu_1, \mu_2,..., \mu_n, S_1,..., S_n)$ where $\mu_1, \mu_2,..., \mu_n$ are the validity of LE_1, LE_2,…, LE_n ($\mu_i \in [0,1]$) and $S_1,..., S_n$ are arbitrary dynamical systems.

The Multiplicative Fuzzy System $\mathbf{f}$ is called Separable Multiplicative Fuzzy System if and only if

- f_i is depended only on μ_i, $\forall i = 1,2,..., n$, but it does not depended on any μ_j $\forall i, j,\ i \neq j$

9 Reciprocal Multiplicative Fuzzy Systems

Suppose the Multiplicative Fuzzy System $\mathbf{f}(\mu_1, \mu_2,..., \mu_n, S_1,..., S_n)$ where $\mu_1, \mu_2,..., \mu_n$ are the validity of LE_1, LE_2,…, LE_n logical sentences ($\mu_i \in [0,1]$) and $S_1,..., S_n$ are arbitrary dynamical systems.

The Multiplicative Fuzzy System $\mathbf{f}$ is called Reciprocal Additive Fuzzy System if and only if

- f_i is dependented on μ_i in the same way that f_j is dependented on μ_j $\forall i, j$

- f_i remains invariant if we interchange the variables $\mu_k, \mu_l \ \forall i,k,l$ with $k \neq i, l \neq i$

10 Differentiable Fuzzy Systems

Suppose that we have the Fuzzy System $\mathbf{f}(\mu_1, \mu_2, ..., \mu_n, S_1, ..., S_n)$ where $\mu_1, \mu_2, ..., \mu_n$ are the values of the validity of LE_1, LE_2,…, LE_n logical sentences ($\mu_i \in [0,1]$) and $S_1, ..., S_n$ are arbitrary dynamical systems.

The Fuzzy System $\mathbf{f}$ is called differentiable of *m-degree* if and only if $\forall i$

- f_i is m-degree differentiable with respect μ_i's. That is that the first m derivatives of f_i with respect to $\mu_1, \mu_2, ..., \mu_n$ exist and are continuous functions of $\mu_1, \mu_2, ..., \mu_n$.

Theorem:
For a reciprocal differentiable system we have:

$$\boxed{\begin{gathered}\frac{\partial f_i}{\partial \mu_i} = \frac{\partial f_j}{\partial \mu_j}, \text{ where } i \neq j \\[2ex] \frac{\partial f_i}{\partial \mu_k} = \frac{\partial f_j}{\partial \mu_l} \text{ where } i \neq j \text{ and } k \neq i \text{ and } l \neq j\end{gathered}}$$

Proof: It is direct application of the definition

Remark: The simple and the general Takagi-Sugeno System are convex and separable fuzzy systems.

11 Examples

Example 11.1
Suppose that we have the rules
Rule 1: If LE_1 then

$$y_1 = 2x_1 + x_2 \tag{4}$$

where $\frac{dx_1}{dt} = -3x_1 + 2x_2 + 2$ and $\frac{dx_2}{dt} = x_1 - 2x_2$ with initial conditions: $x_1 = 1, x_2 = 0$

Rule 2: If LE_2 then

$$y_2 = x_1 + x_2 \tag{5}$$

where $\frac{dx_1}{dt} = -x_1 + 2x_2 + 2$ and $\frac{dx_2}{dt} = x_1 - x_2$ with initial conditions: $x_1 = 1, x_2 = 0$

Suppose that LE_1, LE_2 are dependent on x_1, x_2 as follows:

$\mu_1 = x_1,\ if\ x_1 \in [0,1],\ \mu_1 = 0,\ if\ x_1 < 0$
$\mu_1 = 0,\ if\ x_1 > 1$

$\mu_2 = \frac{x_1 + x_2}{2},\ if\ x_1, x_2 \in [0,1]$,
$\mu_2 = 0,\ f\ x_1 < 0\ or\ x_2 < 0$

Then a reciprocal fuzzy system can be given by the relation:

$$y = \frac{(\mu_1 - \frac{|\mu_1 - \mu_2|}{3})y_1 + (\mu_2 - \frac{|\mu_1 - \mu_2|}{3})y_2}{\mu_1 + \mu_2 - \frac{2|\mu_1 - \mu_2|}{3}}$$

where y_1, y_2 are given from (4) and (5).

Example 11.2
Suppose that we have:
Rule 1: If LE_1 then

$$y_1 = 2x_1 + 3x_2 \tag{6}$$

where $\frac{dx_1}{dt} = -3x_1 + 2x_2 + 2$ and $\frac{dx_2}{dt} = x_1 - 2x_2$ with initial conditions: $x_1 = 1, x_2 = 0$

Rule 2: If LE_2 then

$$y_2 = x_1 - x_2 \tag{7}$$

where $\frac{dx_1}{dt} = -x_1 + 2x_2 + 2$ and $\frac{dx_2}{dt} = x_1 - x_2$ with initial conditions: $x_1 = 1, x_2 = 0$

Then a reciprocal fuzzy system can be given by the relation:

$$y = \frac{f_1(\mu_1, \mu_2)y_1 + f_2(\mu_1, \mu_2)y_2}{f_1(\mu_1, \mu_2) + f_2(\mu_1, \mu_2)}$$

with

$f_1(\mu_1, \mu_2)$=1, if $\mu_1 > 0.9$

$f_1(\mu_1, \mu_2) = \mu_1$, if $\mu_1 < 0.9$ and $\mu_2 \le 0.9$

$f_1(\mu_1, \mu_2) = \mu_1^2$ if $\mu_1 < 0.9$ and $\mu_2 > 0.9$

and

$f_2(\mu_1, \mu_2)$=1, if $\mu_2 > 0.9$

$f_2(\mu_1, \mu_2) = \mu_2$, if $\mu_2 > 0.9$ and $\mu_1 \le 0.9$

$f_2(\mu_1, \mu_2) = \mu_2^2$ if $\mu_2 > 0.9$ and $\mu_1 > 0.9$

where y_1, y_2 are given from (6) and (7)

and μ_1, μ_2 are the validity of LE_1, LE_2 (for example as in Example 11.1).

12 Conclusion

We introduced a general fuzzy model that is based on the Takagi-Sugeno inference engine. We introduced also several interesting concepts like Separable Additive Fuzzy Systems, Reciprocal Additive Fuzzy Systems, Separable Multiplicative Fuzzy Systems, Reciprocal Multiplicative Fuzzy Systems Differentiable Fuzzy Systems. The general fuzzy system of Session 3 can be used for further investigation and application in several branches of science and engineering.

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