# Optimal synthesis of linkages using sensitivity coefficients in path generation problems

SANCIBRIAN, R., GARCIA, P., VIADERO, F. and FERNANDEZ, A.
Department of Structural and Mechanical Design
University of Cantabria
Avda. de los Castros s/n, 39005- Santander, Cantabria
SPAIN

*Abstract:* - In this paper a method for dimensional synthesis in planar mechanism is presented. This method is based on sensitivity analysis, which gives valuable information about the importance of each link. In this way, during the optimization process two main advantages are provided: the use of exact derivatives and a high number of precision points. Furthermore, the easy computation of the sensitivity coefficients is outlined. A four-bar mechanism is used to test the method showing the accuracy and convergence capacity.

*Key-Words: Optimal design, dimensional synthesis, linkages, sensitivity.*

## 1 Introduction

The basic idea of the method presented consists in determining which link or links of the mechanism need the minimum correction to modify the generated path and to approximate it towards the desired path. This is done using the sensitivity analysis of the mechanism. In this work, the term *sensitivity* refers to a coefficient that describes how sensitive one parameter in the mechanism is to a small variation of another parameter. The influence of the sensitivity coefficients in mechanisms has been studied by several authors [1]. However, the sensitivity analyses carried out in all these works have been focused on the determination of the tolerances of the links in order to guarantee the correct manufacture of the mechanism. Moreover, the methodology to obtain analytical derivatives applied to the coupler link has not been obtained in these papers. Applications to optimal synthesis problems require more sophisticated sensitivity analysis to find the appropriate relationship between parameters, permitting the evolution of the problem towards the optimal solution.

The sensitivity is formulated as the first partial derivative of a mathematical expression giving the relationship between the two parameters of interest. So the search of the proposed method is based on the gradient approximation. These sensitivity parameters can be obtained both by means of analytical or numerical derivation. Obviously, analytical derivation has several advantages over using numerical methods [2, 3]. In addition to the higher accuracy, the computational time used is also smaller and this is an important factor in an iterative process, such as optimal synthesis.

## 2 Sensitivity performance

Figure 1 shows a dyad formed by the links $Z_1$ and $Z_2$ in two finitely separated positions, namely $j$ and $j+1$. We consider that the dyad belongs to a larger linkage, which has influence in the kinematic behavior, but its effect will be considered later. Thus, to achieve the necessary motion to pass from position $j$ to $j+1$ it is necessary to supply in the input link an angle increment in the following form: $\Delta\theta_p(j)=\theta_p(j+1)-\theta_p(j)$, where subscript $p$ denotes the number of the input link.

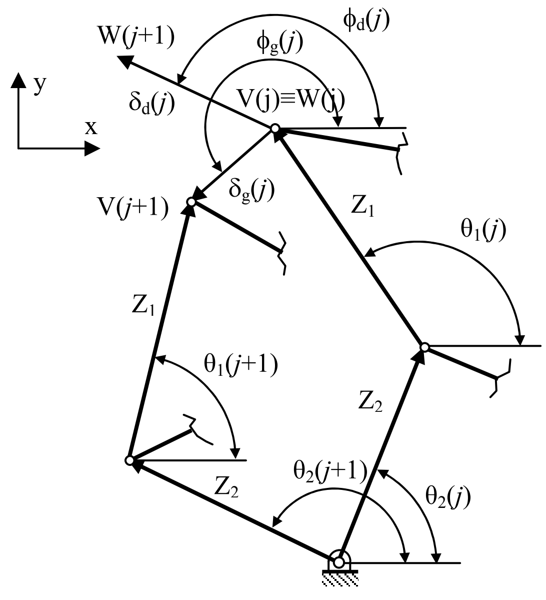



Fig. 1: Dyad notation

In Fig. 1 the point V, defined by the end of the vector $Z_1$, is the point of interest, which means that the path generated by this point during the motion must be controlled in order to achieve the synthesis requirements satisfactorily. In Fig. 1 the generated path between position *j* and *j*+1 is defined using a vector connecting the dyad extremities in both positions. The main objective of the synthesis problem can be summarized as to get the values of the delta vector generated, $\mathbf{\Delta_g}(j)=\delta_g(j)e^{i\phi g(j)}$, closer to the values of the delta vector desired, $\mathbf{\Delta_d}(j)=\delta_d(j)e^{i\phi d(j)}$. The closer the values are, the more accuracy will be achieved by the method.
The mobility range of the input link is limited by external conditions in the problem and, for this reason each value of the $\mathbf{\Delta_d}(j)$ vector has its own $\theta_p(j)$ value assigned to the input link. Thus, to approximate well the $\mathbf{\Delta_g}(j)$ vector to the $\mathbf{\Delta_d}(j)$ requires the modification of some parameters taking part in the kinematic behavior of the linkage. These parameters will be the dimensions of different links and coordinates defined by the vector **Z**. Here, the sensitivity of the vector $\mathbf{\Delta_g}(j)$ with respect to the **Z** elements plays an important role. Thus, the sensitivity is considered for each precision point during the dyad motion, and for each dimension defining the geometry of the linkage. Therefore, a sensitivity matrix is generated whose elements can be expressed as follows,

$$A_k(j) = \frac{\partial \Delta_g(j)}{\partial Z_k} \tag{1}$$

To obtain all these elements it is necessary to establish the relation between the alteration experimented by the coordinates of the point of interest from position $\mathbf{V}(j)$ to $\mathbf{V}(j+1)$ and the mobility range of the links. As is shown in Fig. 1, using complex number theory the loop closure equation describing this relation can be written for one dyad in two finitely separated positions as follows:

$$\mathbf{\Delta_g}(j) = \delta_g(j)e^{i\phi_g(j)} = \sum_{m=1}^{2} Z_m\left(e^{i\theta_m(j+1)} - e^{i\theta_m(j)}\right) \tag{2}$$

In (2) the term containing the summation provides information about the positions and dimensions of the links and their determination is an important aspect in the methodology proposed in this paper. Differentiating this equation with respect to a generic link length, $Z_k$, we obtain,

$$\begin{aligned}\frac{\partial \Delta_g(j)}{\partial Z_k} &= H_k(j) + iG_k(j) = \\ &= \frac{\partial \delta_g(j)}{\partial Z_k} e^{i\phi(j)} + i\frac{\partial \phi_g(j)}{\partial Z_k}\delta_g(j)e^{i\phi(j)}\end{aligned} \tag{3}$$

In (3) the derivatives in the right hand term are the unknown sensitivities that we are looking for. We will call them $A_k(j)$ and $B_k(j)$ respectively. To determine these elements of the sensitivity matrix, this equation can be separated into real and imaginary parts, obtaining a linear system of two equations in two unknowns as follows,

$$\begin{bmatrix} C\phi_g(j) & -\delta_g(j)S\phi_g(j) \\ S\phi_g(j) & \delta_g(j)C\phi_g(j) \end{bmatrix} \begin{Bmatrix} A_k(j) \\ B_k(j) \end{Bmatrix} = \begin{Bmatrix} H_k(j) \\ G_k(j) \end{Bmatrix} \tag{4}$$

where $C\phi_g(j)$ and $S\phi_g(j)$ are $\cos\phi_g(j)$ and $\sin\phi_g(j)$, respectively. In (4) the elements of the matrix are obtained from the dyad motion between consecutive positions. To do that a kinematic analysis is necessary where the $\Delta\theta_p(j)$ is supplied to the input link in the linkage. The vector $\mathbf{H}_k^T=\{H_k(j)\ G_k(j)\}$ is obtained performing the differentiation of the right hand term in (2). In this case, it is necessary to know the rest of the linkage topology and geometry, because it determines the variation of angles $\theta_1$ and $\theta_2$ during the motion and, consequently their derivatives. Expressing (4) in compact form we have,

$$\mathbf{A_k}(j) = [\Phi(j)]^{-1}\mathbf{H_k}(j) \tag{5}$$

where $\mathbf{A_k}(j)=\{A_k(j)\ B_k(j)\}$ will henceforth be called the *sensitivity vector*. Solving this linear system produces,

$$\begin{aligned} A_k(j) &= H_k(j)C\phi_g(j) + G_k(j)S\phi_g(j) \\ B_k(j) &= \frac{1}{\delta_g(j)}\left[H_k(j)C\phi_g(j) - G_k(j)S\phi_g(j)\right] \end{aligned} \tag{6}$$

Equations (6) give the necessary formulation to obtain the elements of the sensitivity matrix shown in (4).
The exact determination of $\mathbf{H}_k(j)^T$ vector is performed by means of the differentiation of the right hand term of (2). After some algebraic manipulation it can be separated into real and imaginary parts and expressed in matricial form,

$$\mathbf{H_k}(j) = [\Delta\theta(j)]\boldsymbol{\delta}_\mathbf{k} + \sum_{m=1}^{2} Z_m[\Delta\theta_m(j)]\mathbf{\Delta\Theta_{mk}}(j) \tag{7}$$

where,

$$[\Delta\theta(j)] = \begin{bmatrix} C\theta_1(j+1) - C\theta_1(j) & C\theta_2(j+1) - C\theta_2(j) \\ S\theta_1(j+1) - S\theta_1(j) & S\theta_2(j+1) - S\theta_2(j) \end{bmatrix} \quad (8)$$

The expression given by (7) is multiplied by the Kroneker delta vector, $\boldsymbol{\delta}_\mathbf{k}^T = \{\delta_{1k}, \delta_{2k}\}$, and

$$[\Delta\theta_m(j)] = \begin{bmatrix} -S\theta_m(j+1) & S\theta_m(j) \\ C\theta_m(j+1) & -C\theta_m(j) \end{bmatrix} \quad (9)$$

The elements of the matrices shown in (8) and (9) are obtained from the kinematic analysis. Therefore, the determination of the elements within the three matrices is immediate. However, problems arise in the determination of the vector $\boldsymbol{\Delta\Theta}_\mathbf{mk}(j)^T$ which is formed by the derivatives of the angles with respect to the linkage dimensions. We call this vector the *angle sensitivity* and it is expressed as follows,

$$\boldsymbol{\Delta\Theta}_\mathbf{mk}(j) = \begin{Bmatrix} \Theta_{mk}(j+1) \\ \Theta_{mk}(j) \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial\theta_m(j+1)}{\partial Z_k} \\ \dfrac{\partial\theta_m(j)}{\partial Z_k} \end{Bmatrix} \quad (10)$$

The aforementioned expressions are relative to the dimension of one link. However, we can expand the matrices to all the dimensions in all links in the mechanism.

In the same way vectors $\mathbf{H}_k(j)$, $\boldsymbol{\delta}_\mathbf{k}$ and $\boldsymbol{\Delta\Theta}_\mathbf{mk}(j)$ are transformed in matrices formed by two rows and $\mathbf{Z}$ columns. Linking (5) with (7) and substituting vectors by matrices we obtain,

$$\mathbf{A}_\mathbf{k}(j) = [\Phi(j)]^{-1}\left([\Delta\theta(j)]\delta + \sum_{m=1}^{2} Z_m [\Delta\theta_m(j)]\boldsymbol{\Delta\Theta}_\mathbf{m}(j)\right) \quad (11)$$

All the formulation must be accomplished with kinematic constraint equations. In general, the design parameters are the dimensions of the links and their positions (i.e. coordinates).

## 3 Linear search method

The first step in the optimization process is to define the *Structural Error Function* to be minimized. First of all, we consider two positions of the linkage where the input link is $Z_2$ and its position has been established previously by the analyst based on synthesis requirements. Thus, the angles $\theta_2(j)$ and $\theta_2(j+1)$ give two finite positions in the linkage related with the positions of the coupler point V($j$) and V($j$+1).

$$\Delta_g^n(j) = \Delta_g\left[\mathbf{Z}_m^n(j), \theta_2(j+1), \theta_2(j)\right] = \delta_g^n(j)e^{i\phi_g^n(j)} \quad (12)$$

From now, the superscript $n$ in (12) is the step iteration. Furthermore, vector $\mathbf{Z}$ modifies its value depending on the position, $j$, and iteration step $n$. In this case the structural error function is formulated using the vectors shown in Fig.1 where $\Delta_g$ is the vector for the generated path and $\Delta_d$ for the desired one. Thus, the function is established using the mean square distance concept as follows,

$$SEF = \frac{1}{2}\sum_{j=1}^{j_{\max}-1}\left\{[\Delta_d^n(j) - \Delta_g^n(j)]^T[\Delta_d^n(j) - \Delta_g^n(j)]\right\} \quad (13)$$

Now, the kinematic synthesis is reduced to the problem of minimizing the objective function given by (13) with respect to the design variables. If we consider only two consecutives precision points the summation is avoided in (13). Differentiating this expression with respect to the $\mathbf{Z}_k$ vector and equating to zero we have,

$$[\mathbf{J}^n(j)]^T[\Delta_d^n(j) - \Delta_g^n(j)] = 0 \quad (14)$$

where the [$J^n$ ($j$)] is the Jacobian matrix. The system of nonlinear equations given by (14) may be solved using a first order approximation.

The Taylor series expansion of (12) with respect to the design variable $Z_k$ and around the values of $Z_k^{n-1}$, $\theta_k^{n-1}(j)$ and $\theta_k^{n-1}(j+1)$ gives the following expression

$$\delta_d(j) \approx \delta_g^{n-1}[\mathbf{Z}^{n-1}(j), \theta_2(j), \theta_2(j+1)] + \left.\frac{\partial\delta_g^n(j)}{\partial Z_k^n(j)}\right|[Z_k^n(j) - Z_k^{n-1}(j)] \quad (15)$$

In equation (15) the term [$Z_k^n(j)$- $Z_k^{n-1}(j)$] represents the dimensional variation that must be experimented by the link $k$ to modify the delta vector modulus to pass from $\delta_g(j)$ to $\delta_d(j)$ using a linear approximation. This approximation can be good enough if the difference between the vectors generated and desired is not large and the absolute value of the modulus vector is small in comparison with the link dimensions. The first derivative in this equation is the sensitivity calculated previously by equation (11). Reordering expression (15) we have,

$$\Delta Z_k^n(j) = Z_k^n(j) - Z_k^{n-1}(j) = \frac{\delta_d^{n-1}(j) - \delta_g^{n-1}(j)}{A_k^{n-1}(j)} \qquad (16)$$

In the same way, a similar mathematical expression can be obtained for the $\phi$ angle,

$$\Delta Z_k^n(j) = Z_k^n(j) - Z_k^{n-1}(j) = \frac{\phi_d^{n-1}(j) - \phi_g^{n-1}(j)}{B_k^{n-1}(j)} \qquad (17)$$

# 4 Four-bar linkage application

The aforementioned procedure can be applied on the four-bar linkage shown in Fig. 2. Here, links $Z_1$ and $Z_2$ or $Z_4$ and $Z_6$ can be considered independently to form the dyad with the point of interest at its extremity. The determination of the sensitivity angle matrices requires special attention and the following matrix must be defined,

$$[\Theta_m(j)] = \begin{bmatrix} \mathbf{\Theta}_{1k}(j) & \mathbf{\Theta}_{(2-5)k}(j) & \mathbf{\Theta}_{6k}(j) \\ \mathbf{\Theta}_{1k}(j+1) & \mathbf{\Theta}_{(2-5)k}(j+1) & \mathbf{\Theta}_{6k}(j+1) \end{bmatrix} \qquad (18)$$

where vector $\mathbf{\Theta}_{(2-5)k}=[\mathbf{\Theta}_{2k}\ \mathbf{\Theta}_{3k}\ \mathbf{\Theta}_{4k}\ \mathbf{\Theta}_{5k}]$ contains a set of sensitivity parameters for both positions: $j$ and $j+1$. Certain dependencies exist between vectors $\mathbf{\Theta}_{1k}(j)$ and $\mathbf{\Theta}_{6k}(j)$ respect to $\mathbf{\Theta}_{3k}(j)$. This relationship is given by the following expressions,

$$\Theta_{m1}(j) = \frac{\partial \alpha_1}{\partial Z_k} + \frac{\partial \theta_3(j)}{\partial Z_k} = \Psi_{1k} + \Theta_{3k}(j) \qquad (19)$$

and

$$\Theta_{m6}(j) = -\frac{\partial \alpha_2}{\partial Z_k} + \frac{\partial \theta_3(j)}{\partial Z_k} = -\Psi_{2k} + \Theta_{3k}(j) \qquad (20)$$

where $\alpha_1$ and $\alpha_2$ are coupler angles and depend on the linkage geometry. To obtain the elements of the matrix shown in (18) a loop closure equation is formulated as follows,

$$\sum_{p=2}^{5} Z_p e^{i\theta_p} = 0 \qquad (21)$$

Equation (21) can be differentiated with respect to the generic dimension of the link $Z_k$ to obtain an equation containing the first order sensitivity angles for k=2,3,4,5. Separating into real and imaginary parts and after some algebraic manipulation it can be expressed in matricial form, we have,

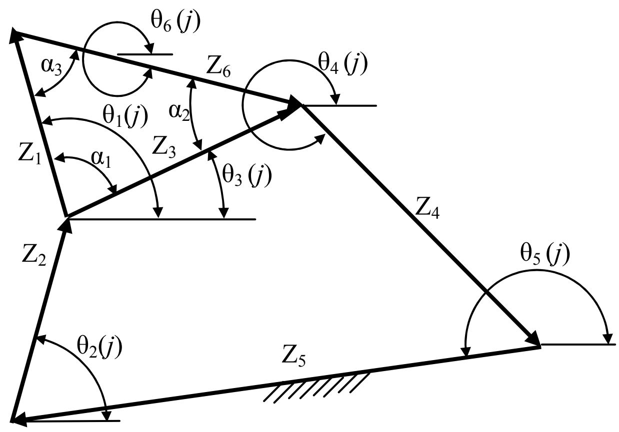



Fig. 2: Four-bar linkage notation

$$[Z_p(j)][\Theta_{(2-5)k}(j)] = -[k(j)] \qquad (22)$$

where

$$[Z_p(j)] = \begin{bmatrix} Z_2C\theta_2(j) & \ldots & Z_5S\theta_3(j) \\ Z_2S\theta_2(j) & \ldots & Z_5C\theta_3(j) \end{bmatrix} \qquad (23)$$

and

$$[k(j)] = \begin{Bmatrix} S\theta_k(j) \\ C\theta_k(j) \end{Bmatrix} \qquad (24)$$

Matrix (23) and vector (24) can be determined by the initial position of the linkage. The $Z_2$ link is the input link and no variation is allowed so it produces the first two sensitivity parameter as follows,

$$\Theta_{2k}(j) = \Theta_{2k}(j+1) = 0 \qquad (25)$$

The second condition refers to the non-motion possibility on the fixed link. This fact produces the same effects as before on its sensitivity coefficient. That is,

$$\Theta_{5k}(j) = \Theta_{5k}(j+1) = 0 \qquad (26)$$

Thus, only two angular sensitivities remain with non-null value for each position. Now it is possible to solve the system shown in (22) and it produces the following expressions,

$$\Theta_{3k}(j) = \frac{\cos[\theta_4(j) - \theta_k(j)]}{Z_3 \sin[\theta_3(j) - \theta_4(j)]} \qquad (27)$$

and

$$\Theta_{4k}(j)=\frac{\cos[\theta_3(j)-\theta_k(j)]}{Z_4\sin[\theta_3(j)-\theta_4(j)]} \tag{28}$$

Following the same procedure for the coupler link, i.e. k=1,6, we have the following closure equation,

$$Z_1e^{i\theta_1(j)}-Z_3e^{i\theta_3(j)}+Z_6e^{i\theta_6(j)}=0 \tag{29}$$

Differentiating this equation with respect to $Z_k$ and solving the linear equation system produces,

$$\Psi_{1k}=\frac{-1}{Z_5\tan\alpha_3} \tag{30}$$

and

$$\Psi_{2k}=\frac{1}{Z_6\sin\alpha_3} \tag{31}$$

Using (30) and (31) together with (19) and (20) the sensitivity coefficients are determined and the linear system of equations given by (22) can be solved. As a consequence sensitivities are obtained from (11) and introduced in the optimization algorithm. This process must be performed for every dimension in every link in all positions.

## 5 Implementation aspects

Since it is a local optimization synthesis method, the starting point must be selected by the analyst using previous experience and intuition. The mechanism is defined using mixed coordinates and the link geometry is not altered during the real motion (kinematic analysis). At this stage, it is also necessary to identify which parameters defining the mechanism will be design variables and they must be included in vector **Z**. Not all the parameters can alter their dimensions in the optimization process, only the design variables can change, all the rest remaining constant. Once the initial guess mechanism is defined, the error must be evaluated at the first couple of precision points. In order to guarantee the convergence of the method it is necessary to minimize, as much as possible, the error in the first couple of precision points by means of the condition $|\Delta_d(1)-\Delta_g^0(1)|\le\varepsilon$. Where $\varepsilon$ is a sufficiently small parameter to guarantee the convergence at the beginning of the process. This condition must be considered in the selection of the initial guess mechanism and does not limit the general application of the method.

Once the initial conditions have been specified, the iterative process starts searching for the optimal solution closest to the starting one. Sensitivity analysis is carried out at every couple of precision points in order to obtain the necessary alteration of the links. Only one link modifies its dimension, or dimensions, to adjust the generated delta vector. The selection of which parameter must be adjusted (modulus or direction) in each position is made based on the minimum link increment. In this way, the modification of the mechanism is always the minimal necessary with respect to the original linkage. The same process is repeated for all points generated by the mechanism until the maximum is achieved. Then, at this point the structural error is evaluated and the convergence criterion applied.

## 6 Examples

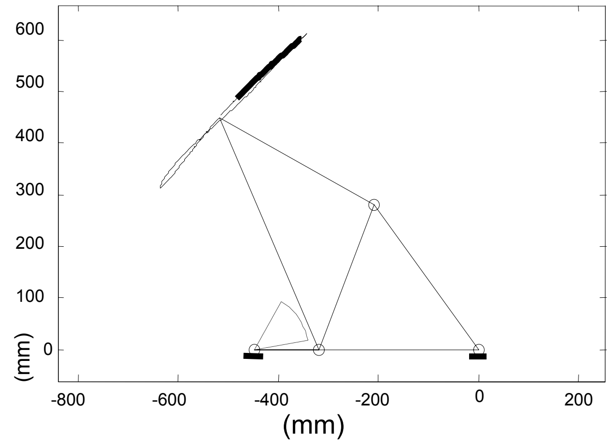
In this example a straight path is selected as the objective, and link 2 is established as the input link. Let us consider the four-bar mechanism shown in Fig. 3a as the initial guess. The following equation gives the desired relation between input link and coupler point position:

$$\left.\begin{aligned}x&=10^2e^{\left(\frac{\theta_2-\theta_o}{55}\right)};\\ y&=0;\end{aligned}\right\} \tag{32}$$

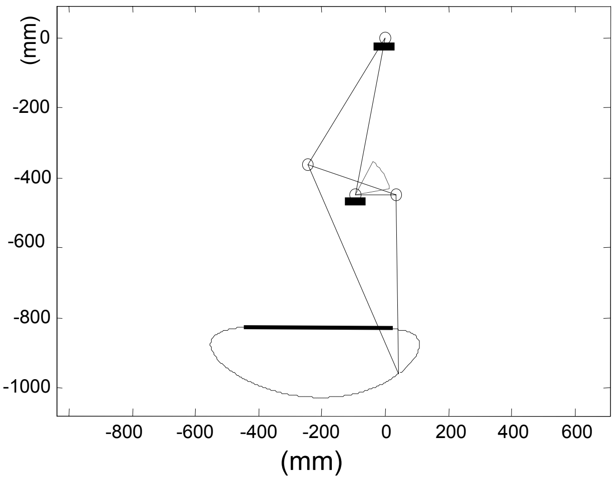
where $\theta_0$ is the initial angle and $\theta_2$ defines the position of the link during the motion. The range of motion of the input link is limited. In Fig 3 the dashed lines, whose apex is on the fixed pair on the input link, shows the motion range. It can be expressed as follows,

$$\theta_2=\left[\frac{\pi}{20},\frac{\pi}{3.5}\right] \tag{33}$$

Eighty equally spaced positions of the input link have been considered within the interval shown in (33), carry out eighty precision points in the desired straight path given by (32). The precision points are not equally spaced and this fact is another constraint that must be solved by the algorithm. Fourteen constraints expressed as algebraic equations are considered for every couple of precision points. The design parameters are the six geometrical dimensions and the fixed angle defining the fixed linkage position. They are shown in the second column in Table 1. The third column in the same table gives the final values of the design parameters. In Fig. 3a the solid line path is the curve generated by the initial guess linkage and synchronized with the input link motion. In the same way, Fig. 3b shows the optimal linkage obtained after 16 iterations. The straight solid line is the final path obtained within the range of interest. The evolution of the global structural error is shown in Fig. 4.




a)




b)

Fig. 3: a) initial guess linkage and b) optimal solution.

Table 1: Initial and optimum values of the design variables.

| Design variables | Initial guess | Optimal solution |
|---|---|---|
| $Z_1$ (mm) | 550.00 | 458.26 |
| $Z_2$ (mm) | 165.00 | 129.93 |
| $Z_3$ (mm) | 96.00 | 295.21 |
| $Z_4$ (mm) | 440.00 | 439.54 |
| $Z_5$ (mm) | 660.00 | 510.01 |
| $Z_6$ (mm) | 472.00 | 662.45 |
| $\theta_5$ (rad) | 3.1416 | 4.5064 |

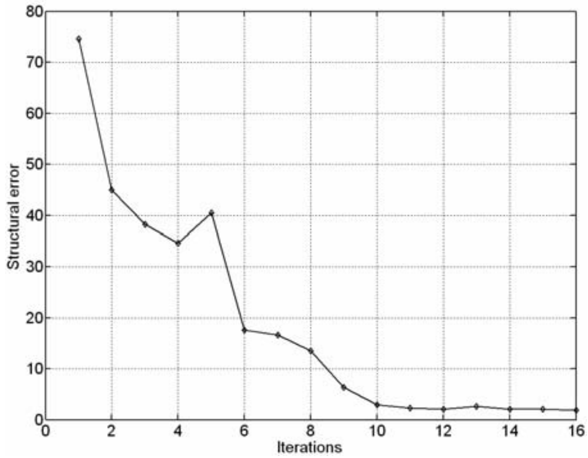



Fig. 4: Global error evolution.

## 7 Conclusions

In this paper we introduce an optimal synthesis method based the use of sensitivity coefficients which leads to a very simple and efficient formulation. Its application to path generation in four-bar linkages shows two main advantages: the use of a large number of precision points and the determination of exact derivatives. Thus, the method is suitable when a high accuracy is demanded by the synthesis problem. Furthermore, the method provides the possibility of correlation between the input link and the path generated by the coupler point in an easy way.

## 8 Acknowledgements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