Processing of Information with Uncertain Boundaries

—Fuzzy Sets and Vague Sets

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Abstract: - In the paper we analyze the relationships and properties of some essential theories between fuzzy sets and vague sets. We find that the uncertain boundaries of the two kinds of sets can be described by a subinterval of [0,1]. Then, the method for measuring the similarity of fuzzy sets is developed based on the idea of vague sets. Finally, a uniform model for measuring the similarity of two same type sets is proposed in each of the two kinds of sets mentioned above. It is proved that fuzzy sets and vague sets are equivalent on measuring the similarity of two same type sets according to this uniform model. And some properties of this model are also presented in this paper. This model is very useful to settle intelligent data information processing and fuzzy control problems.

Key-Words: - Fuzzy sets; Vague sets; Boundary; Membership function; Similarity measure; Degree of similarity

1 Introduction

Fuzzy sets and vague sets are extensions of classical set theory. They can be used in the processing of information with uncertain boundaries [1-3]. The similarity measure of uncertain sets is an important concept of information disposal with uncertain boundaries, and it is also an important basis of processing uncertain information [4-5].

In this paper, we find that all the uncertain boundaries of the two kinds of sets can be described by a subinterval of [0,1]. Following this idea and according to the idea of measuring the similarity of vague sets [5-7], we propose the method of measuring the similarity of fuzzy sets. And then a uniform model for measuring the similarity of two same type sets is presented in each of the two kinds of sets mentioned above. The uniform model satisfies reflexive and symmetric properties et al. It is proved that fuzzy sets and vague sets are equivalent in the uniform model. And the ideas and techniques of this paper can also be easily employed to settle other problems of fuzzy intelligent control and fuzzy clustering analysis.

2 Basic theory

2.1 Basic theory of fuzzy sets

Let U be a non-empty set called the universe, a fuzzy set $\tilde{A}$ of U is defined by a member function $\mu_{\tilde{A}}: U \to [0,1]$. The core and support of fuzzy set $\tilde{A}$ are defined as follows [1]:

$$core(\mu_{\tilde{A}}) = \{u \in U \mid \mu_{\tilde{A}}(u) = 1\}, \quad (1)$$

$$supp(\mu_{\tilde{A}}) = \{u \in U \mid \mu_{\tilde{A}}(u) > 0\}. \quad (2)$$

The boundary region of fuzzy set $\tilde{A}$ is defined as follows:

$$BN_{\tilde{A}} = supp(\mu_{\tilde{A}}) - core(\mu_{\tilde{A}}). \quad (3)$$

2.2 Basic theory of vague sets

Let U be a finite and no-empty universe, $U = \{u_1, u_2, ..., u_n\}$, $\forall u_i \in U$, a vague set $\hat{A}$ in U is characterized by a truth-membership function $t_{\hat{A}}$ and a false-membership function $f_{\hat{A}}$,

$$t_{\hat{A}}: \ U \to [0,1],$$

$$f_{\hat{A}}: \ U \to [0,1],$$

where $t_{\hat{A}}(u_i)$ is a lower boundary on the grade of membership of u_i derived from the evidence for u_i, $f_{\hat{A}}(u_i)$ is a lower boundary on the negation of u_i derived from the evidence against u_i, $t_{\hat{A}}(u_i) + f_{\hat{A}}(u_i) \le 1$ [2]. The grade of membership of u_i in the vague set $\hat{A}$ is bounded to a subinterval $[t_{\hat{A}}(u_i), 1 - f_{\hat{A}}(u_i)]$ of $[0,1]$.

Definition 1 Let $\hat{A}$ be a vague set of universe U, the positive region of $\hat{A}$ is defined as follows:

$$\hat{A}_{-} = \{u_i \mid u_i \in U, t_{\hat{A}}(u_i) = 1\}.$$

Definition 2 Let $\hat{A}$ be a vague set of universe U, the upper approximation of $\hat{A}$ is defined as follows:

$$\hat{A}^{-} = \{u_i \mid u_i \in U, f_{\hat{A}}(u_i) < 1\}.$$

Definition 3 Let $\hat{A}$ be a vague set of universe U, the boundary region of $\hat{A}$ is defined as follows:

$$BN_{\hat{A}} = \hat{A}^{-} \setminus \hat{A}_{-} = \{u_i \mid u_i \in U, t_{\hat{A}}(u_i) < 1 \wedge f_{\hat{A}}(u_i) < 1\}.$$

2.3 Relations among $\mu_{\tilde{A}}(x)$, $t_{\hat{A}}(x)$ and $f_{\hat{A}}(x)$

Fuzzy set theory or vague set theory emphasizes the membership relations of elements in a set, namely, membership degree. The membership functions of fuzzy sets and vague sets are obtained by the statistics or experiences of experts.

All membership function values of a fuzzy set are a single value of $[0,1]$. This single value contains only two kinds of information of an element x ($x \in U$), i.e., support degree and negative degree. Whereas, all membership function values of a vague set are a subinterval of $[0,1]$. This subinterval contains three kinds of information of an element x ($x \in U$), i.e., support degree, negative degree and unknown degree. Vague sets are more accurate to describe some vague information than fuzzy sets.

Theorem 1 Let A be a subset of the universe of discourse U, for $\forall x \in U$. Then in fuzzy sets and vague sets, the grade of membership of x in A could be described by a subinterval $[a,b]$ of $[0,1]$, where $a, b \in [0,1]$, a is a lower boundary on the grade of membership of x derived from the evidence for x in A, and b is a upper boundary on the grade of membership of x derived from the evidence for x in A.

Proof. In fuzzy set A, the grade of membership of x in A is $\mu_A(x)$, it is used to create a subinterval $[a,b]$ of $[0,1]$, namely, $[a,b]=[\mu_A(x),\mu_A(x)]$. In vague set A, the grade of membership of x in A is bounded by a subinterval $[t_{\hat{A}}(x),\ 1-f_{\hat{A}}(x)]$ of $[0,1]$, namely, $[a,b]=[t_{\hat{A}}(x),\ 1-f_{\hat{A}}(x)]$.

Theorem 2 In fuzzy sets and vague sets, their uncertainties of boundary regions can be described with the lower approximation subtracted from the upper approximation.

Proof. In fuzzy set $\tilde{A}$, the lower approximation of set $\tilde{A}$ is

$$\tilde{A}_{-}=\{u\in U\mid \mu_{\tilde{A}}(u)=1\},$$

and the upper approximation of set $\tilde{A}$ is

$$\tilde{A}^{-}=\{u\in U\mid \mu_{\tilde{A}}(u)>0\},$$

then the boundary region of set $\tilde{A}$ is $\tilde{A}^{-}\setminus\tilde{A}_{-}$.

In vague set $\hat{A}$, by using Definition 3, the boundary region of set $\hat{A}$ is $\hat{A}^{-}\setminus\hat{A}_{-}$.

3 Similarity measures between two same type sets of fuzzy sets and vague sets

3.1 Similarity measure between vague sets [5-7]

Let $\hat{A}$ and $\hat{B}$ be two vague sets in the universe of discourse U, $U=\{u_1,u_2,...,u_n\}$, $\forall u_i\in U$, $a_i=[t_{\hat{A}}(u_i),1-f_{\hat{A}}(u_i)]$ be the vague membership value of u_i in vague set $\hat{A}$, and $b_i=[t_{\hat{B}}(u_i),$ $1-f_{\hat{B}}(u_i)]$ be the vague membership value of u_i in vague set $\hat{B}$. Then the degree of similarity between the vague values a_i and b_i can be evaluated by function M as follows:

$$M(a_i,b_i)=1-\frac{|S(a_i)-S(b_i)|+|t_{\hat{A}}(u_i)-t_{\hat{B}}(u_i)|+|f_{\hat{A}}(u_i)-f_{\hat{B}}(u_i)|}{4},\qquad (4)$$

where $S(a_i)=t_{\hat{A}}(u_i)-f_{\hat{A}}(u_i)$ and

$$S(b_i)=t_{\hat{B}}(u_i)-f_{\hat{B}}(u_i)\ (1\le i\le n).$$

Let vague sets $\hat{A}$ and $\hat{B}$ be defined as follows:

$$\hat{A}=[t_{\hat{A}}(u_1),1-f_{\hat{A}}(u_1)]/u_1+[t_{\hat{A}}(u_2),1-f_{\hat{A}}(u_2)]/u_2+...+[t_{\hat{A}}(u_n),1-f_{\hat{A}}(u_n)]/u_n,$$

$$\hat{B}=[t_{\hat{B}}(u_1),1-f_{\hat{B}}(u_1)]/u_1+[t_{\hat{B}}(u_2),1-f_{\hat{B}}(u_2)]/u_2+...+[t_{\hat{B}}(u_n),1-f_{\hat{B}}(u_n)]/u_n,$$

Then the degree of similarity between the vague sets $\hat{A}$ and $\hat{B}$ can be evaluated by function $VSimD$ as follows:

$$VSimD(\hat{A},\hat{B})=\begin{cases}1, & if\ \hat{A}=\hat{B}=\Phi;\\ 0, & else\ if\ condition;\\ \dfrac{1}{n}\sum_{i=1}^{n}M(a_i,b_i), & otherwise.\end{cases}\qquad (5)$$

where *condition* is $t_{\hat{A}}(u_i) = 0$ and $t_{\hat{B}}(u_i) \neq 0$, or $t_{\hat{A}}(u_i) \neq 0$ and $t_{\hat{B}}(u_i) = 0$, at the same time,

$$t_{\hat{A}}(u_i) + f_{\hat{A}}(u_i) = 1 \text{ and}$$

$$t_{\hat{B}}(u_i) + f_{\hat{B}}(u_i) = 1 \ (i = 1,2,...,n).$$

$M(a_i, b_i)$ is defined in Eq.(4).

3.2 Similarity measure between fuzzy sets

Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy sets in the universe of discourse U, $U = \{u_1, u_2, ..., u_n\}$, $\forall u_i \in U$, $a_i = \mu_{\tilde{A}}(u_i)$ be the fuzzy membership value of u_i in set $\tilde{A}$, and $b_i = \mu_{\tilde{B}}(u_i)$ be the fuzzy membership value of u_i in $\tilde{B}$. Then the degree of similarity between the fuzzy values a_i and b_i can be evaluated by function M as follows:

$$M(a_i, b_i) = 1 - |\mu_{\tilde{A}}(u_i) - \mu_{\tilde{B}}(u_i)|, \qquad (6)$$

where $1 \leq i \leq n$.

Let fuzzy sets $\tilde{A}$ and $\tilde{B}$ of the universe of discourse U be defined as follow:

$$\tilde{A} = \mu_{\tilde{A}}(u_1)/u_1 + \mu_{\tilde{A}}(u_2)/u_2 + \ldots + \mu_{\tilde{A}}(u_n)/u_n,$$

$$\tilde{B} = \mu_{\tilde{B}}(u_1)/u_1 + \mu_{\tilde{B}}(u_2)/u_2 + \ldots + \mu_{\tilde{B}}(u_n)/u_n.$$

Then the degree of similarity between the fuzzy sets $\tilde{A}$ and $\tilde{B}$ can be evaluated by function $FSimD$ as follows:

$$FSimD(\tilde{A}, \tilde{B}) = \begin{cases} 1, & if \quad A = B = \Phi; \\ 0, & else \ if \ condition; \\ \frac{1}{n}\sum_{i=1}^{n} M(a_i, b_i) & otherwise, \end{cases} \qquad (7)$$

where *condition* is $\mu_{\tilde{A}}(u_i) \neq 0$ and $\mu_{\tilde{B}}(u_i) = 0$, or $\mu_{\tilde{A}}(u_i) = 0$ and $\mu_{\tilde{B}}(u_i) \neq 0$ $(i = 1,2,...,n)$.

$M(a_i, b_i) = 1 - |\mu_{\tilde{A}}(u_i) - \mu_{\tilde{B}}(u_i)|$, $a_i = \mu_{\tilde{A}}(u_i)$ and $b_i = \mu_{\tilde{B}}(u_i)$.

3.3 A uniform model of the similarity measure between two same type sets of fuzzy sets and vague sets

Based on Theorem 1 and the similarity measures of fuzzy sets and vague sets, we propose a uniform model for measuring the similarity of two same type sets in each of the two kinds of sets mentioned above. The uniform model is defined as follows:

$$SimD(A, B) = \begin{cases} 1, & if \ A = B = \Phi; \\ 0, & else\,if \ condition; \\ \frac{1}{n}\sum_{i=1}^{n} M(a_i, b_i), & otherwise, \end{cases} \qquad (8)$$

where U is a finite and no-empty set called the universe, $U = \{u_1, u_2, ..., u_n\}$. A and B are two same type sets of U. The *condition* in the Eq.(8) is $t_A(u_i) = 0$ and $t_B(u_i) \neq 0$, or $t_A(u_i) \neq 0$ and $t_B(u_i) = 0$, at the same time, $t_A(u_i) + f_A(u_i) = 1$ and $t_B(u_i) + f_B(u_i) = 1$. The degree of similarity between sets A and B can be evaluated by function $SimD(A, B)$. In fuzzy sets, $SimD(A, B)$ is $FSimD(A, B)$. In vague sets, $SimD(A, B)$ is $VSimD(A, B)$. The $M(a_i, b_i)$ in the Eq.(8) is defined as follows:

$$M(a_i, b_i) = 1 - \frac{|S(a_i) - S(b_i)| + |t_A(u_i) - t_B(u_i)| + |f_A(u_i) - f_B(u_i)|}{4},$$

(9)

where the degree of similarity about element u_i between sets A and B is evaluated by $M(a_i, b_i)$. In $M(a_i, b_i)$, $a_i = [t_A(u_i), 1 - f_A(u_i)]$ and $b_i = [t_B(u_i), 1 - f_B(u_i)]$ (by applying Theorem 1), $S(a_i) = t_A(u_i) - f_A(u_i)$, $S(b_i) = t_B(u_i) - f_B(u_i)$, thereinto:

If A is a vague set then $t_A(u_i) = t_A(u_i)$ and $f_A(u_i) = 1 - (1 - f_A(u_i))$.

If A is a fuzzy set then $t_A(u_i) = \mu_A(u_i)$ and $f_A(u_i) = 1 - \mu_A(u_i)$.

$t_B(u_i)$ and $f_B(u_i)$ are similar to $t_A(u_i)$ and $f_A(u_i)$.

Obviously, these measure methods of fuzzy sets and vague sets are equivalent on measuring the similarity of two same type sets in this uniform model Eq. (8). The larger the value of $SimD(A, B)$, the more the similarity between the sets A and B. The smaller the value of $SimD(A, B)$, the less the similarity between the sets A and B. This uniform model Eq. (8) satisfies some properties as follows:

Theorem 3 In fuzzy sets or vague sets, $M(a_i, b_i)$ satisfies some properties:

(1) $M(a_i, b_i) \in [0,1]$,

(2) $M(a_i, b_i) = M(b_i, a_i)$,

(3) $M(a_i, b_i) = 0 \Leftrightarrow a_i = [0,0]$ and $b_i = [1,1]$, or $a_i = [1,1]$ and $b_i = [0,0]$, where $1 \le i \le n$.

The proof is omitted.

Theorem 4 In fuzzy sets or vague sets,

$$SimD(A, B) \in [0,1].$$

Proof. The conclusion is clear according to Theorem 3 and Eq. (8).

Theorem 5 In fuzzy sets or vague sets, $SimD(A, B)$ satisfies

(1) reflexive: $SimD(A, A) = 1$;

(2) symmetric: $SimD(A, B) = SimD(B, A)$;

Theorem 6 In fuzzy sets or vague sets,

$SimD(A, B) = 0$ if and only if $t_A(u_i) = 0$ and $t_B(u_i) \neq 0$, or $t_A(u_i) \neq 0$ and $t_B(u_i) = 0$,

meanwhile, $t_A(u_i) + f_A(u_i) = 1$ and

$t_B(u_i) + f_B(u_i) = 1$ for all $i = 1,2,..., \mathrm{n}$.

Where

$$a_i = [t_A(u_i), 1 - f_A(u_i)],$$

$$b_i = [t_B(u_i), 1 - f_B(u_i)].$$

Proof. The conclusion is a direct result of Eq. (8).

For any subsets $A, B \subseteq U$, $SimD(A, B)$ doesn't satisfy the transitivity.

4 Conclusion

In this paper, we analyze these relations between some concepts of fuzzy sets and vague sets, and present that the uncertain boundaries of the two kinds of sets can be described by a subinterval of [0,1]. Then, the method for measuring the similarity of fuzzy sets is developed based on the idea of vague sets. Based on the Theorem 1, we proposed a uniform model for measuring the similarity of two same type sets of fuzzy ones or vague ones. This uniform model satisfies reflexive and symmetric properties et al. It is proved that fuzzy sets and vague sets are equivalent on measuring the similarity of two same type sets in this uniform model. This model is very useful to handling uncertain reasoning and fuzzy intelligent control problems.

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