Instantaneous Frequency Estimation of FM Signals Using the T - Class of Time-Frequency Distributions

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Abstract:- Recently we have proposed the T-class of time-frequency distributions (TFD's) with time-only kernels to provide high-resolution and considerable cross-terms reduction for multicomponent FM signals. In this work we investigate the instantaneous frequency (IF) properties of two members of this class: the hyperbolic and the exponential T-distributions in the presence of noise. A comparison with two well-known TFD's, Wigner-Ville and Choi-Williams distributions, is also considered.

Keywords: Time-frequency analysis, Wigner-Ville distribution, instantaneous frequency

1 Introduction

Non-parametric instantaneous frequency (IF) estimation for multicomponent non-stationary signals is an important issue in signal processing [1, ?, 2]. The concept of the instantaneous frequency can be found in [1, 3, 4, 5]. Time-frequency analysis is used for IF estimation for multicomponent signals as it is the only reliable tool to reveal the multicomponent nature of such signals by concentrating the signal energy in the time-frequency plane around the component IF laws [3]. These energy concentrations are known as "peaks" or "ridges" of the time-frequency representation or distribution (TFD). However, quadratic time-frequency distributions of multicomponent signals suffer from the presence of cross-terms [3, 4, 5], which can obscure the real features of interest in the signal. Considerable efforts have been made to define TFD's which reduce the effect of cross-terms while improving the time-frequency resolution [3, 5]. However, there is always a compromise between these two requirements. TFD's have different performances in this respect and the choice of the proper TFD is application dependent.

A class of time-frequency distributions with high time-frequency resolution and strong cross-terms reduction was proposed in [1, 7] and proved to be effective for both mono- and multicomponent FM signals. Members of this class has kernels that are functions of time only. We shall refer to these TFD's with time-only kernels as the T-distributions (TD's). In this paper we show that this class is also efficient in IF estimation of mono- and multicomponent FM signals in the presence of additive gaussian noise. Its performance is compared to two widely used members of the quadratic class of TFD's: The Choi-Williams Distribution (CWD) and the Wigner-Ville Distribution (WVD).

2 The IF Concept and TFD's

For time-frequency analysis of a real signal $x(t)$, we always consider its analytic associate $z(t) = x(t) + j\hat{x}(t)$, where $\hat{x}(t)$ is the Hilbert transform of $x(t)$ [4].

Consider an analytic signal of the form

$$z(t) = ae^{j\phi(t)} + \epsilon(t) \tag{1}$$

where the amplitude a is constant, and $\epsilon(t)$ is a complex-valued white Gaussian noise with independent identically distributed (i.i.d.) real and imaginary parts with total variance σ_ϵ^2. The instantaneous frequency of $z(t)$ is given by

$$f_i(t) = (1/2\pi)d\phi(t)/dt \tag{2}$$

We assume in this analysis that $f_i(t)$ is an arbitrary, smooth and differentiable function of time with bounded derivatives of all orders.

The general equation for quadratic time-frequency representation of a signal $z(t)$ is given by [4]

$$\rho(t,f) = \underset{\tau\to f}{\mathcal{F}}[G(t,\tau) \underset{(t)}{*} K_z(t,\tau)] \tag{3}$$

where $G(t,\tau)$ is the time-lag kernel, $K_z(t,\tau) = z(t+\tau/2)z^*(t-\tau/2)$ and $\underset{(t)}{*}$ denotes time convolution. The kernel could also be expressed in the Doppler-lag domain as $g(\nu,\tau$, where

$$G(t,\tau) = \underset{\nu\to t}{\mathcal{F}^{-1}}\{g(\nu,\tau)\} \tag{4}$$

In the discrete lag domain $\rho(t, f)$ will be

$$\rho(t,f) = \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} K_z(u, 2mT) \times G(t-u, 2mT) e^{-j4\pi f mT} du \quad (5)$$

where m is an integer and T is the sampling interval. If $\rho(t, f)$ is discretized over time and frequency then we have

$$\rho(n,k) = \sum_{l=-N_s}^{Ns-1} \sum_{m=-N_s}^{N_s-1} K_z(lT, 2mT) \times G(nT-lT, 2mT) e^{-j2\pi \frac{km}{2N_s}} \quad (6)$$

where $2N_s$ is the number of samples. The frequency samples are given by $f_k = k/4N_sT$.

The IF estimate is a solution of the following optimization

$$\widehat{f}_i(t) = \arg[\max_f \rho_h(t,f)] \quad ; \; 0 \le f \le f_s/2 \quad (7)$$

where $f_s = 1/T$ is the sampling frequency. The frequency estimation error is the difference between the actual value in eq.(2) and the estimate in eq.(5) as follows

$$\Delta\widehat{f}_i(t) = f_i(t) - \widehat{f}_i(t) = \phi'(t)/2\pi - \widehat{f}_i(t). \quad (8)$$

Therefore, the bias and variance of this estimate will be

$$\begin{aligned} B(\hat{f}_i(t)) &= \mathcal{E}[\Delta\hat{f}_i(t)] = f_i(t) - \mathcal{E}[\hat{f}_i(t)] \\ V(\hat{f}_i(t)) &= \mathcal{E}[\Delta\hat{f}_i(t)]^2 = \mathcal{E}[\{f_i(t) - \hat{f}_i(t)\}^2] \quad (9) \end{aligned}$$

As we will see later, this bias is zero for single-tone and linear FM (LFM) signals, and therefore a Cramer-Rao bound (CRB) exists for the variance.

3 The T-Distributions

Time-only kernels are a special case of separable time-lag kernels. Suppose we have a separable time-lag kernel as follows

$$G(t,\tau) = g_1(t) g_2(\tau) \quad (10)$$

where g_1 and g_2 are continuous and L^2 integrable functions.

It was shown in [7] that for best time-frequency resolution we should have

$$\begin{aligned} G(t,\tau) &= \mathrm{G}(t) = g_1(t)/M \\ g(\nu,\tau) &= \mathrm{g}(\nu) = \mathop{\mathcal{F}^{-1}}_{t\to\nu}\{g_1(t)\}/M \quad (11) \end{aligned}$$

where $G(t,\tau)$ is now a time-only kernel. This is the formula for all time-only kernels, which are the kernels of the T-distributions.

The Exponential T-Distribution (ETD): the kernel of the Choi-Williams distribution (CWD) in the Doppler - lag domain is $g(\nu,\tau) = \exp(-4\pi^2\nu^2\tau^2/\sigma)$ which can be given in the time-lag domain by [4]

$$G(t,\tau) = \sqrt{\sigma/4\pi\tau^2} \exp(-\sigma t^2/4\tau^2) \quad (12)$$

where σ is a real parameter. In [7], we proposed a time-frequency distribution $T_e(t, f)$ with the following exponential *time-only kernel*

$$G(t,\tau) = \mathrm{G}_\sigma(t) = \sqrt{\sigma/\pi} \exp(-\sigma t^2) \quad (13)$$

where σ is a real parameter and $\sqrt{\sigma/\pi}$ is a normalization factor. It was shown in [1] that the resolution of the ETD exceeds that of CWD by far.

The hyperbolic T-distribution (HTD): it has the following time-only kernel [7]

$$G(t,\tau) = \mathrm{G}_\sigma(t) = k_\sigma / \cosh^{2\sigma}(t) \quad (14)$$

where σ is a real positive number and k_σ is a normalization factor given by

$$k_\sigma = \int_{-\infty}^{\infty} 1/\cosh^{2\sigma}(t) = \Gamma(2\sigma)/2^{2\sigma-1}\Gamma^2(\sigma)$$

in which Γ represents the gamma function.

4 IF Using TD's

It can be shown that the T-distributions do not satisfy the time marginal property, hence they do not satisfy the traditional condition for the instantaneous frequency. But in [1] we proposed the following general IF property: at any time t, the time-frequency distribution $\rho(t, f)$ should have absolute maximum at $f = \frac{1}{2\pi}\frac{d\phi(t)}{dt}$, which is the actual important characteristic needed for IF estimation. In [1] we have shown that at any t, the hyperbolic T-distribution *has absolute maximum* at $f = \frac{1}{2\pi}\frac{d\phi(t)}{dt}$ for linear FM signals. This is general for all T-distributions and constitutes the basis for our IF estimation. For non-linear FM signals this IF estimate is biased, and best IF estimation is achieved in this case by adaptive methods [1]. For an FM signal of the form $z(t) = a\ e^{j\phi(t)}$, a being a constant, the general formula for the T-distributions can be given by

$$\begin{aligned} \rho_z(t,f) &\approx |a|^2 \int \mathrm{G}(t-u)\, \delta\, [\frac{1}{2\pi}\phi'(u) - f]\, du \\ &= |a|^2 \mathrm{G}(t-\psi(f))\psi'(f) \quad (15) \end{aligned}$$

where ψ is the inverse of $\frac{1}{2\pi}\phi'$, i.e., $\frac{1}{2\pi}\phi'(\psi(f)) = f$ and it is assumed that there is a relatively small effect from higher-order derivatives $\phi^{(k)}(t), k \geq 3$. Assuming that $\psi^{'}(f)$ is not a highly peaked function of f and knowing that $\mathrm{G}(t - \psi(f))$ is peaked at $t = \psi(f)$ since it is low-pass and even in t, the absolute maximum of $\rho_z(t, f)$ for any time t would be at $\psi(f) = t$, or $f = \frac{1}{2\pi}\phi^{'}(t)$, which is the instantaneous frequency of the FM signal $z(t)$. For nonlinear FM signals, the energy peak of $\rho_z(t, f)$ is actually biased from the instantaneous frequency due to the higher-order phase derivatives. The major contribution in this term is due to $\phi^{(3)}(u)$ [1]. Therefore at the instants of rapid change in the IF law the bias is not negligible and eq.(14) would not be an accurate approximation to $\rho_z(t, f)$ unless suitable windowing in the lag direction is used. An adaptive window length would be recommended, but due to significant bias no CRB is applicable.

For linear FM (LFM) signals we have $\phi^{(k)}(t) = 0$ for $k \geq 3$. Assuming $z(t) = ae^{j2\pi(f_o t + \frac{\beta_o}{2}t^2)}$, where f_o and β_o are constants, we have

$$d(t, f) = \frac{1}{\beta_o} \mid a \mid^2 \mathrm{G}_\alpha\Big(t - \frac{1}{\beta_o}(f - f_o)\Big) \qquad (16)$$

which has an absolute maximum at $f = f_o + \beta_o t$, the instantaneous frequency of the linear FM signal $z(t)$. As $\beta_o \rightarrow 0$, i.e., $z(t)$ approaches a sinusoid, we have $d(t, f) \rightarrow \mid a \mid^2 \delta(f - f_o)$, in accordance with eq.(4) for a monocomponent single-tone signal.

5 Simulation Results

The above time-frequency distributions were simulated and the IF was estimated according to eqs. (1) and (6-9). A linear FM signal of length $N = 512$ samples was selected and i.i.d noise samples were added for different SNR's. The sampling frequency was $f_s = N$ Hz. For each SNR, 1000 Monte Carlo iterations were considered for the purpose of calculating the variance of the IF estimate. Fig.(1) shows the result of this simulation for three TFd's. The performance of the HTD is distinguished as superior to other TFD's, especially at low SNR's. Performance of the ETD is nearly to that of the HTD in this particular example.

6 Conclusion

This paper has shown that the recently developed T-Class of time-frequency distributions (TFD's) outperforms other well - known distributions like the Wigner - Ville distribution (WVD) and the Choi - Williams distribution (CWD) in terms of robustness. The HTD gave a minimal variance for all SNR's, however, the difference in performance is more evident for low SNR's, where the T-Class distributions outperform other TFD's BY FAR.

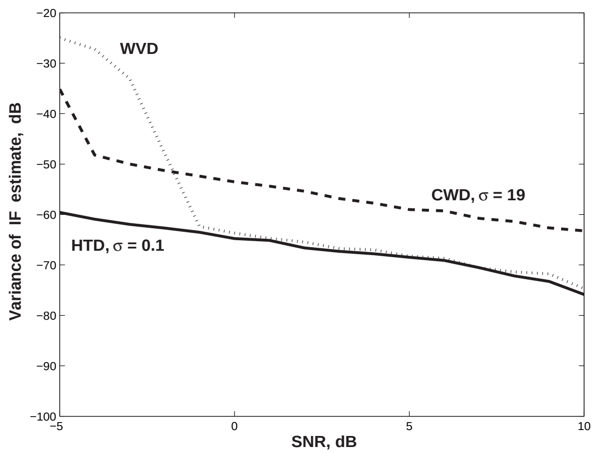

Figure 1: Performance of various TFD's in IF estimation of a linear FM signal with length $N = 512$ samples. The sampling frequency was $f_s = N$ Hz. It is evident that the recently proposed HTD surpasses other TFD's in robustness where it gives the minimum variance, especially at low SNR's.

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