Metric on a Statistical Space-Time

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Abstract: - We introduce a concept of distance for a space-time where the notion of point is replaced by the notion of physical states e.g. probability distributions. We apply ideas of information theory and compute the Fisher information matrix on such a space-time. This matrix is the metric on that manifold. We apply these ideas to a simple model and show that the Lorentzian metric can be obtained if we assumed that the probability distributions describing space-time fluctuations have complex values. Such complex probability distributions appear in non-Hermitian quantum mechanics.

Keywords: - Distance, Entropy, Relative Entropy, Fisher information metric, space-time

1 Introduction

The concept of distance between two points plays a central role in any physical theory. This notion is well defined as long as space-time can be regarded as a classical system. Fundamental theories of nature are quantum theories, i.e. the fields describing the particles and forces have to be quantized or in other words replaced by operators. This procedure is often referred to as second quantization. Whereas this procedure is well understood for Yang-Mills theories that are relevant to describe the electroweak and strong interactions, it is far less clear how to quantize and how to renormalize Einstein’s theory of general relativity that describes gravitation. One thus expects that general relativity has to be modified in the high energy regime. This might be a hint that space-time structure is more complicated at very short distances.

One common speculation is that at energies where fluctuations of the metric become relevant space-time becomes fuzzy. There is then an uncertainty in the measurement of a length, see e.g. [13] for a recent review. It has been shown that quantum mechanics and general relativity considered together imply the existence of a minimal length in nature [3]:

\[ \delta x \geq \ell_p \]  

where \( \ell_p \) where is the Planck length.

We will make a different and more radical assumption. It has been proposed long ago by Rosen [17] that the notion of point might not be relevant anymore at short distances or equivalently at high energies. The basic assumption is that a physical point is not a well localized entity but is fuzzy in the sense that the only information one has is of statistical nature, namely that the “mathematical” point is localized within a certain volume. We substitute the notion of a four dimensional point \( x^\mu \) by the notion of a distribution \( \bar{x}^\mu = p_{\theta^\mu}(x^\mu) \). Such an assumption is not that different from the one made in noncommutative geometry where points \( x^\mu \) are replaced by noncommutative operators \( \hat{x}^\mu \) [20]. One can imagine different concepts for distance. For example, in the case of noncommutative geometry, Connes’ distance [6] can be used. The question we want to address is the following: how can a distance be defined if the notion of a well localized point is replaced by the notion of a distribution? It turns out that ideas from information theory can be applied. We will propose a definition for the distance on such a statistical space-time. This paper is organized as follows: we will first review the concept of Shannon entropy and explain how it leads to the introduction of Fisher’s metric. We will then apply these ideas to a simple model for a fluctuating space-time, define a distance and compute the metric on the manifold of distributions. We then conclude.
2 Brief review of Fisher information metric

There are many excellent reviews and books on Fisher information metric, a nice introduction can be found in [8]. A distance \( d(P_1, P_2) \) between two points \( P_1 \) and \( P_2 \) has to satisfy the following three axioms:

1. Positive definiteness: \( \forall P_1, P_2 : d(P_1, P_2) \geq 0 \)
2. Symmetry \( d(P_1, P_2) = d(P_2, P_1) \)
3. Triangle inequality: \( \forall P_1, P_2, P_3 : d(P_1, P_2) \leq d(P_1, P_3) + d(P_3, P_2) \).

This concept of distance can be traced back to Aristotle and Euclid.

It is often useful to introduce a concept of distance between elements of a more abstract set. For example, one could ask what is the distance between two distributions between e.g. the Gaussian and binomial distributions. It is useful to introduce the concept of entropy as a mean to define distances. In information theory, Shannon entropy [19] represents the information content of a message or, from the receiver point of view, the uncertainty about the message the sender produced prior to its reception. It is defined as

\[
-\sum_{i} p(i) \log p(i),
\]

where \( p(i) \) is the probability of receiving the message \( i \). The unit used is the bit. The relative entropy can be used to define a “distance” between two distributions \( p(i) \) and \( g(i) \). The Kullback-Leibler [10] distance or relative entropy is defined as

\[
D(g||p) = \sum_{i} g(i) \log \frac{g(i)}{p(i)}
\]

where \( p(i) \) is the real distribution and \( g(i) \) is an assumed distribution. Clearly the Kullback-Leibler relative entropy is not a distance in the usual sense: it satisfies the positive definiteness axiom, but not the symmetry or the triangle inequality axioms. It is nevertheless useful to think of the relative entropy as a distance between distributions.

The Kullback-Leibler distance is relevant to discrete sets. It can be generalized to the case of continuous sets. For our purposes, a probability distribution over some field (or set) \( X \) is a distribution \( p : X \in \mathbb{R} \), such that

1. \( \int_X d^x p(x) = 1 \)
2. For any finite subset \( S \subset X \), \( \int_S d^x p(x) > 0 \).

We shall consider families of distributions, and parameterize them by a set of continuous parameters \( \theta \) that take values in some open interval \( M \subset \mathbb{R}^4 \). We use the notation \( p_\theta \) to denote members of the family. For any fixed \( \theta \), \( p_\theta : x \mapsto p_\theta(x) \) is a mapping from \( X \) to \( \mathbb{R} \). We shall consider the extension of the family of distributions \( F = \{ p_\theta \theta \in M \} \), to a manifold \( M \) such that the points \( p \in M \) are in one to one correspondence with the distributions \( p \in F \). The parameters \( \theta \) of \( F \) can thus be used as coordinates on \( M \).

The Kullback number is the generalization of the Kullback-Leibler distance for continuous sets. It is defined as

\[
I(g_\theta||p_\theta) = \int d^x g_\theta(x) \log \frac{g_\theta(x)}{p_\theta(x)},
\]

Let us now study the case of an infinitesimal difference between \( q_\theta(x) = p_{\theta+\epsilon v}(x) \) and \( p_\theta(x) \):

\[
I(p_{\theta+\epsilon v}||p_\theta) = \int d^x p_{\theta+\epsilon v}(x) \log \frac{p_{\theta+\epsilon v}(x)}{p_\theta(x)}
\]

Expanding in \( \epsilon \) and keeping \( \theta \) and \( v \) fix one finds (see e.g. [15]):

\[
I(p_{\theta+\epsilon v}||p_\theta) = I(p+\epsilon||p)|_{\epsilon=0} + \epsilon I'(\epsilon)|_{\epsilon=0} + \frac{1}{2} \epsilon^2 I''(\epsilon)|_{\epsilon=0} + O(\epsilon^3).
\]

One finds \( I(0) = I'(0) = 0 \) and

\[
I''(0) = v^\mu \int_X d^x p_\theta(x) \left( \frac{1}{p_\theta(x)} \frac{\partial p_\theta(x)}{\partial \theta^\mu} \right) \left( \frac{1}{p_\theta(x)} \frac{\partial p_\theta(x)}{\partial \theta^\nu} \right) v^\nu.
\]

We can now identify the Fisher information metric [7] on a manifold of probability distributions as

\[
g_{\mu\nu} = \int_X d^x \frac{1}{p_\theta(x)} \frac{\partial p_\theta(x)}{\partial \theta^\mu} \frac{\partial p_\theta(x)}{\partial \theta^\nu}.
\]
It has been show that this matrix is a metric on a manifold of probability distributions, see e.g. [16]. Corcuera and Giummolé [5] have shown that the Fisher information metric is invariant under reparametrization of the sample space $X$ and that it is covariant under reparametrizations of the manifold, i.e. the parameter space, see e.g. [21] for a review. Fisher’s information matrix plays an important role in many different fields. This concept appears in such different fields as e.g. instanton calculus [22], ontology [2] or econometrics [11]. Symbolic computations of Fisher information matrices have also been considered [14].

3 Fisher information metric and distance on fluctuating spaces

Let us now apply the ideas developed in the previous chapter to a simple model of space-time. Let us assume that the notion of points $x^\mu$ is replaced by a state $\bar{x}$ that could be for example a distribution $p_{\theta^\mu}(x^\mu)$. We propose the following definition

$$I(q_{\theta^\mu}(x^\mu)||p_{\theta^\mu}(x^\mu)) = \int d^4x q_{\theta^\mu}(x^\mu) \log \frac{q_{\theta^\mu}(x^\mu)}{p_{\theta^\mu}(x^\mu)}$$

for the distance between two “points” $p_{\theta^\mu}(x^\mu)$ and $q_{\theta^\mu}(x^\mu)$. The metric on the manifold of distributions is given locally by

$$g_{\mu\nu} = \int d^4xp_{\theta}(x) \left( \frac{1}{p_{\theta}(x)} \frac{\partial p_{\theta}(x)}{\partial x^\mu} \right) \left( \frac{1}{p_{\theta}(x)} \frac{\partial p_{\theta}(x)}{\partial x^\nu} \right)$$

and corresponds to the Fisher information matrix. The distance between two points $A^\mu$ and $B^\nu$ on the manifold is given by $d(A^\mu, B^\nu) = \sqrt{g_{\mu\nu}A^\mu B^\nu}$.

As a example, one can consider for example a 3-dimensional Gaussian distribution

$$p_{\theta}(x) = \frac{1}{(2\pi a^2)^3} e^{-\frac{(x-a_1)^2+(y-a_2)^2+(z-a_3)^2}{2a^2}}$$

the Fisher metric reads $g_{ij} = 1/a^2 \text{diag}(1, 1, 1)$, note that one has the freedom to rescale the relative entropy by a factor $a^2$, it wish case it is simply the matrix $\text{diag}(1, 1, 1)$. The Fisher information matrix as already been calculated in the literature for a Gaussian distribution, see e.g. [4], where the parameter $a$ was interpreted as $\theta_0$, in our case we choose to treat $a$ as scale parameter of the model and not to identify it with a coordinate on the manifold.

An interesting question arises: can we generate a four dimensional space-time with a Lorentzian signature $\text{diag}(-1, 1, 1, 1)$? One has to solve the following system of equations:

$$g_{00} = \int d^4x \frac{1}{p_{\theta}(x)} \left( \frac{\partial p_{\theta}(x)}{\partial \theta^0} \right)^2 = -1$$
$$g_{ii} = \int d^4x \frac{1}{p_{\theta}(x)} \left( \frac{\partial p_{\theta}(x)}{\partial \theta^i} \right)^2 = 1 \text{ for } i \in \{1, 2, 3\}$$
$$g_{\mu\nu} = \int d^4x \frac{1}{p_{\theta}(x)} \left( \frac{\partial p_{\theta}(x)}{\partial \theta^\mu} \right) \left( \frac{\partial p_{\theta}(x)}{\partial \theta^\nu} \right) = 0 \text{ for } \mu \neq \nu.$$  

The first of these equations $g_{00} = -1$ is not solvable if $p_{\theta}(x)$ is a real probability distribution as the ones usual consider in quantum mechanics. In that case $g_{00}$ is always positive and $(\frac{\partial p_{\theta}(x)}{\partial \theta^\mu})^2$ is also positive. One way out is to extend the definition we gave of a probability distribution to include complex probability distributions. This is not as surprising as it might sound. Non-Hermitian quantum mechanics has been used to deal with physical phenomena involving metastable finite-lifetime states, so-called resonances [12] and for the study of delocalization phenomena such as bacteria populations, vortex spinning in superconductors or hydrodynamical problems [9]. In non-Hermitian quantum mechanics density probabilities are complex functions [1]. The complex transition probability is a measurable quantity in e.g. electron-quantum dot scattering-like experiments [18].

As an ansatz, let us consider the complex probability distribution:

$$p_{\theta}(x) = \frac{1}{(2\pi a^2)^2} e^{-\frac{(x-a_1)^2+(y-a_2)^2+(z-a_3)^2}{2a^2}}.$$

This distribution is normalized: $\int d^4xp_{\theta}(x) = 1$ and leads to the following Fisher information matrix

$$(g_{\mu\nu}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$  

(14)
after a rescaling by \( a^2 \). It is interesting to note that a complex Gaussian distribution can lead to a Lorentzian metric. This might be a hint that the Lorentzian metric is due to a non-Hermitian nature of quantum gravity, whatever that theory might be. In our case the signature of space-time and in particular the special nature of time can be related to an imaginary parameter \( i\theta \). The Fisher information metric can be seen as a statistical average over space-time fluctuation. The macroscopic Lorentzian metric appears as a consequence of the statistical distribution of space-time points.

### 4 Conclusions

In this paper we have considered a new concept of distance for a space-time where points are replaced by states that can be for example distributions or operators. We apply ideas developed in information theory, the Fisher information is the metric on the manifold of states. We define the distance between two vectors on this manifold. We apply this new concept to a simple model for a fluctuating space-time. We show that if we extend the domain of definition of the probability distributions from real to complex numbers, we can recover a Lorentzian metric.

### References


