The Dynamics of Fuzzy Cellular Automata: Rule 30

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Abstract

We continue the investigation into the dynamics and evolution of fuzzy rules, obtained by the fuzzification of the disjunctive normal form, and initiated for Rule 90 in [3] and continued for Rule 110 in [5]. We present new results regarding the dynamics of fuzzy Rule 30 whose Boolean evolution is known to be chaotic [7, p.871]. In particular, we show that in the fuzzy case with a finite support configuration all temporal sequences are aperiodic, and their convergence is strongly dependent upon their positions along key diagonals. It follows that fuzzy Rule 30 is neither chaotic nor random. It turns out that the evolution and dynamics in this case differ radically from those in fuzzy Rule 90.

1. INTRODUCTION

This work is motivated by a question posed by Andy Wuensche of Santa Fe [8] regarding the convergence of fuzzy rules. He asked whether the results in [3] for fuzzy rule 90 presented in the conference cited in [8] could be extended to fuzzy rule 30 in the generality obtained in [3]. The interest in Rule 30 cannot be underestimated; for instance, its Boolean counterpart is actually used as a random number generator in Wolfram’s Mathematika, [7]. In this paper we answer this question in a somewhat mysterious form: Although some of the results regarding fuzzy rule 90 are similar to those obtained in this case (e.g., the aperiodicity of temporal sequences for finite support configurations), they differ dramatically in general as we show. For a brief introduction to fuzzy cellular automata see [1], [2], [3]. Furthermore, we develop some general methods for obtaining information about any one of the 255 fuzzy rules, thus unifying their underlying dynamics to a large extent. We refer the reader to [3] and [5] for basic terminology.

Definition 1 A Fuzzy CA is obtained by fuzzification of the local function of a Boolean CA: in the disjunctive normal form, \((a \lor b)\) is replaced by \((a + b)\), \((a \land b)\) by \((ab)\), and \((-a)\) by \((1 - a)\). The resulting local rule is a real-valued function simulating the original function on \([0, 1]^3\), with \(l(a, 0) = 1 - a\) and \(l(a, 1) = a\):

\[g : [0, 1]^3 \rightarrow [0, 1]\]

s.t. \(g(x_1, x_2, x_3) = \sum_{i=0}^{7} r_i \prod_{j=1}^{3} l(x_j, d_{ij})\).
The usual fuzzification of the expression \(a \lor b\) is \(\max\{1, a + b\}\) so as to ensure that the result is not larger than 1. Note, however, that taking \((a + b)\) for the CA fuzzification does not lead to values greater than 1 since the sum of all the expressions for Rule 255 is 1 (i.e., \(g_{255}(x, y, z) = 1\)), and so every (necessarily non-negative) partial sum must be bounded by 1. Since every fuzzy rule is obtained by adding one or more of these partial sums it follows that every fuzzy rule is bounded below by 0 and above by 1.

Example 2 Consider rule 14 = 2 + 2\(^2\) + 2\(^3\):

\[
(000, 001, 010, 011, 100, 101, 110, 111) \rightarrow (0, 1, 1, 1, 0, 0, 0, 0).
\]

The canonical (boolean) expression of rule 14 is: (see [6] for further details)

\[
g_{14}(x_1, x_2, x_3) = (\neg x_1 \land \neg x_2 \land x_3) \\
\lor (\neg x_1 \land x_2 \land \neg x_3) \lor (\neg x_1 \land \neg x_2 \land \neg x_3).
\]

The fuzzification process after simplification yields:

\[
g_{14}(x_1, x_2, x_3) = (1 - x_1) \cdot (x_2 + x_3 - x_2 \cdot x_3).
\]

Example 3 Rule 18 = 2 + 2\(^4\) has the local rule

\[
(000, 001, 010, 011, 100, 101, 110, 111) \rightarrow (0, 1, 0, 0, 1, 0, 0, 0).
\]

Its canonical expression being

\[
g_{18}(x_1, x_2, x_3) = (\neg x_1 \land \neg x_2 \land x_3) \lor \\
\neg x_1 \land x_2 \land \neg x_3,
\]

we obtain its fuzzification as

\[
g_{18}(x_1, x_2, x_3) = (1 - x_2)(x_1 + x_3 - 2x_1 \cdot x_3). \tag{1}
\]

As a final example we cite the rule that motivated this paper:

Example 4 Rule 30 = 2 + 2\(^2\) + 2\(^3\) + 2\(^4\) has the local rule

\[
(000, 001, 010, 011, 100, 101, 110, 111) \rightarrow (0, 1, 1, 1, 0, 0, 0, 0)
\]

Its canonical expression being

\[
g_{30}(x_1, x_2, x_3) = (\neg x_1 \land \neg x_2 \land x_3) \\
\lor (\neg x_1 \land x_2 \land \neg x_3) \\
\lor (x_1 \land \neg x_2 \land \neg x_3)
\]

its fuzzification turns it into

\[
g_{30}(x_1, x_2, x_3) = x_1 + x_2 + x_3 \tag{2}
\]

To begin with, we fix the notation, see [3]. The light cone from a cell \(x_i^t\) is the set of \(\{x_j^{t+p} \mid p \geq 0 \text{ and } j \in \{i - p, \ldots, i + p\}\}\). In this case, the light cone is the boundary of an infinite triangle whose vertex is at the singleton \(a\) and whose boundary consists of all the other \(a\’s\). Thus, \(x_{\pm n}^m\) will denote the cell at \(\pm n\) steps to the right/left of the zero state at time \(m\). For example, \(x_{-1}^2 = x_3^2 = 7/16\) in Table 1. The single cell \(x_0^0\) will be denoted by \(a\) and generally we will take it that \(0 < a \leq 1\), since the case \(a = 0\) is clear.

It is easy to see that the only fixed points of \(g_{30}\) that fall inside \(U\) are zero (a repelling fixed point) and one-half (an attracting fixed point). Starting from a single value \(a\) in a zero background, the spatio-temporal evolution of fuzzy Rule 30 is represented in Table 1 below:
Table 1: Rule 30: Evolution from a single point \( a \) in a zero background.

<table>
<thead>
<tr>
<th>Time</th>
<th>(-3)</th>
<th>(-2)</th>
<th>(-1)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>(\ldots)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(\ldots)</td>
<td>0</td>
<td>0</td>
<td>(\frac{1}{4})</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>(\ldots)</td>
</tr>
<tr>
<td>1</td>
<td>(\ldots)</td>
<td>0</td>
<td>0</td>
<td>(\frac{1}{4})</td>
<td>(\frac{1}{4})</td>
<td>(\frac{1}{4})</td>
<td>0</td>
<td>(\ldots)</td>
</tr>
<tr>
<td>2</td>
<td>(\ldots)</td>
<td>0</td>
<td>(\frac{1}{4})</td>
<td>(\frac{7}{16})</td>
<td>(\frac{15}{32})</td>
<td>(\frac{3}{8})</td>
<td>(\frac{1}{4})</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>(\ldots)</td>
<td>(\frac{1}{4})</td>
<td>(\frac{37}{128})</td>
<td>(\frac{615}{1024})</td>
<td>(\frac{1067}{2048})</td>
<td>(\frac{257}{512})</td>
<td>(\frac{7}{16})</td>
<td>(\frac{1}{4})</td>
</tr>
<tr>
<td>4</td>
<td>(\ldots)</td>
<td>(\frac{175}{256})</td>
<td>(\frac{873}{1536})</td>
<td>(\frac{3031}{3031})</td>
<td>(\frac{2401}{2401})</td>
<td>(\frac{4116}{4116})</td>
<td>(\frac{8187}{8187})</td>
<td>(\frac{15}{16})</td>
</tr>
<tr>
<td>(\vdots)</td>
<td>(\vdots)</td>
<td>(\vdots)</td>
<td>(\vdots)</td>
<td>(\vdots)</td>
<td>(\vdots)</td>
<td>(\vdots)</td>
<td>(\vdots)</td>
<td>(\vdots)</td>
</tr>
</tbody>
</table>

From this simple example, we see that the dynamics of this rule will be far from obvious and will certainly not parallel those of Rule 90 in [3]. Indeed, we do not expect convergence of diagonals to the common limit of \(1/2\), in general, even though this is an attracting fixed point.

Referring to Example 4 we know that the canonical expression of this rule is given by (2). We note that \(x_n^m = a\) for \(n = 0, 1, 2, \ldots\) and so clearly, \(L_0^-(a) = \lim_{n \to -\infty} x_n^m = a\). The number, \(L_0^-(a)\) will denote the limit of the zeroth left-diagonal. In general, the quantity \(L_n^-(a)\) will denote the limit of the \(n\)th left-diagonal (whenever this limit exists), \(n = 0, 1, 2, \ldots\).

Remark 5 There is no loss of generality in assuming that the \(k\)th diagonal consists only of cells of the form \(x_{-m}^n\), with \(m \geq 0\) since the remaining cells, those that lie up and to the right, make up a finite set and thus cannot affect the convergence properties of this sequence.

The remainder of this section is devoted to the following theorem on the long-term dynamics of fuzzy Rule 30.

**Theorem 6** Let \(a \in (0, 1)\) be a single seed in a zero background. Then the dynamics of fuzzy Rule 30 are given as follows: For \(k \geq 0\), the \((k+1)\)th left-diagonal converges to \(L_{k+1}^- = L_{k+1}^-(a)\) where \(L_0^-(a) = a\), \(L_1^-(a) = 1\) and

\[
L_{k+1}^- = \frac{L_{k+1}^- + L_k^- - 2L_{k+1}^- L_k^-}{2L_{k+1}^- + L_k^- - 2L_{k+1}^- L_k^-}.
\]

The right-diagonal \(L_0^+(a) = a\) and all remaining right-diagonals converge to \(L_k^+(a) = 1/2\). All vertical columns converge to \(1/2\).

**Proof.** The next or first left diagonal (below the zeroth) consists of the elements \(x_{-n}^{n+1}\), \(n = 0, 1, 2, \ldots\). Observe that the dynamics of this specific diagonal are governed by iterations of a rule of the form \(g(0, y, z) = y + z - yz\) (this is essentially fuzzy Rule 14 with \(x_1 = 0\), see Example 2 above). Now, \(y + z - yz = 1 - (1 - y)(1 - z)\). Therefore, \(x_{-1}^2 = g(0, a, a) = 1 - (1 - a)^2\). Similarly, we can show that \(x_{-2}^3 = g(0, a, x_{-1}^2) =\)
$1 - (1 - a)^3$. A simple induction argument on $n$ shows that $x_{n+1} = 1 - (1 - a)x_n$ for any given choice of $a \in (0, 1)$. It follows that $L_1^-(a) = \lim_{n \to \infty} x_{n+1}^n = 1$.

In order to find the limit, $L_2^-(a)$, we use the local function for this rule along with the fact that $L_1^-(a) = 1$. This gives $L_2^-(a) = 1 - a$. In general it is easy to see using a lengthy but simple induction argument that

$$L_{k+1}^- = \frac{L_{k-1}^- + L_k^- - 2L_{k-1}^- L_k^-}{2L_{k-1}^- + L_k^- - 2L_{k-1}^- L_k^-}, \quad (3)$$

where (3) can be thought of as a nonlinear three-term recurrence relation for the $L_k^-$, $k \geq 0$, with initial conditions $L_0^-(a) = a$, $L_1^-(a) = 1$. We note that (3) defines a rational function of $a$ because of the nature of the rule. Furthermore, we observe that $\deg L_{k+1}^- = \deg L_k^- + \deg L_{k-1}^-$. Since $\deg L_1^- = 0$ and $\deg L_2^- = 1$, we see that $\deg L_{k+1}^-$ must be the Fibonacci number, $F_k$. The first few limits are displayed below:

$$L_3^- = \frac{a}{1 + a},$$
$$L_4^- = \frac{a^2 - a + 1}{a^2 - 2},$$
$$L_5^- = \frac{a^3 - a^2 - 1}{a^3 - 2a - 1},$$
$$L_6^- = \frac{-(2a + 1 + 2a^2 - 3a^2 - 2a^4 + a^5)}{-3a - 2 - 3a^3 + 4a^2 + a^4}, \ldots$$

**Remark 7** Note that even though each limit $L_k^-(a)$ is generally not 1/2, the limit of the limits is actually 1/2 (compare with fuzzy Rule 110 in [5]). By this we mean that the sequence of limits $L_k^-(a)$, converges to 1/2 as $k \to \infty$ regardless of the value of the seed, $a$! The reason for this follows from the fact that if $L_k^-(a) \to L(a)$ as $k \to \infty$, then the same is true for $L_{k-1}^-(a)$ and $L_{k+1}^-(a)$. It follows from (3) that such a limit is given by the solution

$$L = \frac{2L - 2L^2}{3L - 2L^2} \quad (4)$$

The only solution of this equation that lies in $[0, 1]$ is $L = 1/2$. Clearly $L = 0$ is not a solution since the right hand side of (4) converges to 2/3 while the left side converges to 0.

The same methods can now be applied to the right-diagonals, $S_{m+}^r$, where $m \geq 0$, emanating from the seed $x_0^m = a$ starting with the zeroth right diagonal ($S_0^r$) that must consist only of $a$’s (recall that the boundary of the light cone starting at $a$ consists of only of $a$’s and thus $L_0^+(a) = a$). On the other hand, the first right diagonal, denoted by the set $S_1^r$, is governed by iterations of the function $g(x, y, 0) = x + y - 2xy$ (the canonical expression for fuzzy Rule 102). Once again, a moment’s notice shows that $L_1^+(a) = 1/2$.

We use the local function above to get subsequent limits: Thus $L_2^+(a) = 1/2$. Similarly, one can conclude that $L_{k+1}^+(a) = 1/2$ for $k = 0, 1, ...$. In order to find the limits of the right-vertical sequences of the form ${x_{m+p}^r}_{p=0}^\infty$, $m = 1, 2, 3, \ldots$, we proceed as indicated in the general situation above: The union of two sets of points each of which has exactly one common point of accumulation also has exactly one (and the same) point of accumulation. In this case the right-half of the infinite cone, $C^+$, whose vertex is at $x_1^0 = a$ is the countable union of sets of (right-diagonal) sequences each of which converges to 1/2 and so 1/2 is the only point of accumulation of $C^+$. Since right-vertical sequences are infinite subsequences of $C^+$, we get that all such sequences (or columns) must have the point 1/2 as their only limit point.

The case of the asymptotic behavior of the right-vertical sequences must be considered separately, due to the existence of an infinite number of limit points lying in the left-half
of the light-cone with vertex at a. For these left-vertical sequences (denoted by $V_i^-$) we note that it suffices to show that the first left-vertical sequence, $V_1^-$ has a limit equal to one-half. For the central vertical sequence $V_0$ converges to 1/2 from the preceding paragraph. Furthermore, the first right-vertical sequence, $V_1^+$, also converges to 1/2. Using the local function it follows that $x_{-1}^{m-1} \to 1/2$. The same argument applied to $V_2$ and its two successors $V_1^-, V_0$ gives that $x_{-2}^{m-1} \to 1/2$. A simple induction argument on the column number (that we leave to the reader) gives the result that all left-vertical columns (or sequences) converge to 1/2. This completes the proof.

An illustration of this phenomena using the seed $a = 0.17$, for calculation purposes only, follows in Table 3:

<table>
<thead>
<tr>
<th>Time</th>
<th>Local states</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0 0 0 .17 0 0 0 ...</td>
</tr>
<tr>
<td>1</td>
<td>0 0 .17 .17 .17 0 0 ...</td>
</tr>
<tr>
<td>2</td>
<td>0 .17 .3111 .37533 .2822 .17 0 ...</td>
</tr>
<tr>
<td>3</td>
<td>.17 .42821 .54598 .51950 .476119 .35625 .17 ...</td>
</tr>
<tr>
<td>4</td>
<td>.52542 .65866 .54047 .47717 .49365 .49836 .40513 ...</td>
</tr>
</tbody>
</table>

Table 2: Rule 30: Evolution from the point $a = 0.17$ in a zero background.

We compare this fuzzy view of Rule 30 with its Boolean counterpart below, [4]. If one draws an imaginary line down the central axis of the light cone in Table 4 we note a pattern on the right reminiscent of the patterns observed in Rule 90 (cf., [3]). These patterns indicate convergence of the fuzzy rule to 1/2. The variety of patterns on the left side of the light cone is indicative of the infinitely many different limits observed in the dynamics of the left-diagonals of fuzzy rule 30.

**Concluding Remarks** Even though it is well known that Boolean rule 30 is random or chaotic, this is not the case for fuzzy rule 30. It also appears that, in general, all temporal sequences are aperiodic at best. Minor modifications can be made here in order for the techniques to apply to neighborhood structures with an arbitrary fixed number of cells (e.g., 5 or 7) and even finite support initial configurations in these instances.
<table>
<thead>
<tr>
<th>Time</th>
<th>...</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>...</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>.83</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>...</td>
</tr>
<tr>
<td>1</td>
<td>...</td>
<td>0</td>
<td>0</td>
<td>.83</td>
<td>.83</td>
<td>.83</td>
<td>0</td>
<td>0</td>
<td>...</td>
</tr>
<tr>
<td>2</td>
<td>...</td>
<td>0</td>
<td>.83</td>
<td>.9711</td>
<td>.18907</td>
<td>.2822</td>
<td>.83</td>
<td>0</td>
<td>...</td>
</tr>
<tr>
<td>3</td>
<td>...</td>
<td>.83</td>
<td>.995</td>
<td>.18547</td>
<td>.57734</td>
<td>.73505</td>
<td>.64375</td>
<td>.83</td>
<td>...</td>
</tr>
<tr>
<td>4</td>
<td>...</td>
<td>.99916</td>
<td>.17264</td>
<td>.34580</td>
<td>.74409</td>
<td>.43726</td>
<td>.29342</td>
<td>.40512</td>
<td>...</td>
</tr>
<tr>
<td>...</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3: Rule 30: Evolution from the point $a = 0.83$ in a zero background.

Table 4: Boolean Rule 30, - Evolution from a single point in a zero background

References


[4] from mathworld.wolfram.com


