Admissible Solution of an $N$-periodic Leontief Dynamic Model

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Abstract: In this work we consider the $N$-periodic Leontief dynamic model and we analyze some conditions over the coefficient matrices in order to characterize when the solution of the system is nonnegative. To solve this problem descriptor system, positive control and some topics related to $N$-periodic control theory are used. Nonnegative matrices play an important role in these results.

Key-Words: Leontief dynamic model, nonnegative matrix, positive solution, descriptor system.

1 Introduction

Since 1949 when Leontief developed an economic model that described the interrelationships among different sectors in the United States economy, a number of different approaches have been used to the extension of input-output models. The dynamic model extends the static model and it can describe the real situation more faithfully. Recently, the control theory is one area that has attracted interest in the study of economic models. The methods of control can be used to solve economic problems and it is worthwhile to use dynamic control systems to analyze the evolution of the variables in the economic model. Normally, the discrete-time approach seems the most suitable to describe this kind of model. In the study of this model the nonnegative matrices play an important role.

In this work, we analyze a model with two different compartimental sectors at time $k$: the first includes only the star products and the second the uninteresting products. But, the situation changes periodically because the situation of star good can be changed to uninteresting good or new goods can be introduced. We analyze some conditions over the coefficient matrices in order to characterize the positiveness of the system.

Consider the $N$-periodic Leontief model described by the descriptor system

$$C(k)x(k + 1) = (I - P(k) + C(k))x(k) - D(k)u(k),$$

where the matrices $C(k+N) = C(k)$, $P(k+N) = P(k)$, $D(k+N) = D(k)$, and $C(k)$ is the capital coefficient matrix at time $k$, $P(k)$ is the technological coefficient matrix at time $k$, $D(k)$ is the demand coefficient matrix (excluding investment) at time $k$, $x(k)$ is the production level vector and $u(k)$ is the demand level vector.

Since the Leontief model is an economic model, $C(k) \geq 0$, $P(k) \geq 0$ and $D(k) \geq 0$ that is, all the entries of these matrices are nonnegative. This kind of system is a dynamic descriptor system because the capital coefficient matrix $C(k)$ can be nonsingular. The entries of the coefficient capital matrix are the required capital per unit of production per sector. The singularity of this matrix
arises because no output from one sector is used in the production of some products. By the nature of the technological matrix \( P(k) \), it is possible to choose \( p_{ij}(k) \), such that \( 0 \leq p_{ij}(k) < 1 \), and

\[
\sum_{i=1}^{n} p_{ij}(k) < 1, \quad j = 1, \ldots, n.
\]

In this work we suppose that the matrices \( P(k) \) satisfy the above condition. Then, the Leontief matrices \( (I - P(k)) \) are invertible, and

\[
\det((\lambda C(k)) - (I - P(k) + C(k))) = \det(-I + P(k)) \neq 0
\]

for \( \lambda = 1 \). Thus, the system (1) has a solution, see [2]. But unfortunately, some components of this solution can be negative. This means that the solution is not good for the economic problem. When the positiveness of the solution of the system cannot be assured, it is worthwhile studying if from an initial nonnegative state and under an appropriate control sequence it is possible to obtain a nonnegative trajectory of the solution. When all the components of the solution the system (1) are nonnegative we said that the system has an admissible solution.

In particular, for the invariant case, a initial characterization for positive descriptor systems is given in [1]. Some results about Leontief dynamics models have been given in [3, 4, 5].

2 Periodic economic model

Consider the \( N \)-periodic Leontief dynamic system

\[
C(k)x(k + 1) = (I - P(k) + C(k))x(k) - D(k)u(k).
\]

Premultiplying the system (2) by \( (I - P(k))^{-1} \), we have

\[
E(k)x(k + 1) = A(k)x(k) - \mathbf{D}(k)u(k)
\]

where

\[
E(k) = (I - P(k))^{-1} C(k),
\]

\[
A(k) = (I + E(k))
\]

and

\[
\mathbf{D}(k) = (I - P(k))^{-1} D(k).
\]

Note that, \( E(k), A(k) \) and \( \mathbf{D}(k) \) are nonnegative matrices.

Suppose that there exists an \( N \)-periodic collection of permutation matrices, \( \{Q(k), k = 0, 1, \ldots, N - 1\} \), such that by the state transformation

\[
\tilde{x}(k) = Q(k) x(k),
\]

we obtain a similar system \( \left( \tilde{E}(\cdot), \tilde{A}(\cdot), \tilde{D}(\cdot) \right)_N \), where

\[
\tilde{E}(k) = Q(k + 1) E(k) Q^{-1}(k + 1)
\]

\[
\tilde{A}(k) = Q(k + 1) A(k) Q^{-1}(k)
\]

\[
\tilde{D}(k) = Q(k + 1) \mathbf{D}(k)
\]

with \( E_1(k) \) an invertible matrix and \( N(k) \) a nilpotent matrix. Note that, matrices \( I_{n_2} + N(k) \), \( k = 0, 1, \ldots, N - 1 \) are invertible matrices.

The system \( \left( \tilde{E}(\cdot), \tilde{A}(\cdot), \tilde{D}(\cdot) \right)_N \) means that the model has two different compartmental sectors at time \( k \): the first includes only the star products and the second the uninteresting products. This does that the capital matrix has a submatrix nilpotent, because the uninteresting situation have to finish in certain period of time.

Note that, this model only represents two kind of products, the star and the uninteresting one but they can be change for a period of time, because the situation of star good can be changed to uninteresting good or new goods can be introduced.
Then, the capital and the technological matrices can be variables in the time. In this work we are considering \(N\)-periodic changes.

The new \(N\)-periodic Leontief dynamic model is given by

\[
\begin{bmatrix}
E_1(k) & O \\
O & N(k)
\end{bmatrix}
\bar{x}(k+1) = \begin{bmatrix}
\Phi_1(k) \\
\Phi_2(k)
\end{bmatrix}
\bar{x}(k) - \begin{bmatrix}
D_1(k) \\
D_2(k)
\end{bmatrix} u(k)
\]

The first step is to obtain the solution. The state solution of this system gives a lot of information, because it informs on that future input production is needed to supply the future market demand.

Given the system (4) and premultiplying this equation by

\[
\begin{bmatrix}
E_1^{-1}(k) & O \\
O & (I_n + N(k))^{-1}
\end{bmatrix}
\]

we have the following equivalent system

\[
\begin{bmatrix}
I_n & 0 \\
0 & (I_n + N(k))^{-1}
\end{bmatrix}
\bar{x}(k+1) = \begin{bmatrix}
E_1^{-1}(k) & I_n \\
O & I_n
\end{bmatrix}
\bar{x}(k) - \begin{bmatrix}
E_1^{-1}(k)D_1(k) \\
(I_n + N(k))^{-1}D_2(k)
\end{bmatrix} u(k).
\]

This dynamic model includes the stocks and flows of capital goods explicitly.

To obtain the solution we introduce the following matrices

\[
\Phi_1(k, k_0) = \prod_{i=k_0}^{k-1} \left( E_1^{-1}(i) + I_n \right) \begin{cases} I_n & k > k_0 \\ O & k < k_0 \end{cases},
\]

and

\[
\Psi_2(k, k_0) = \prod_{i=k}^{k_0} \left( I_n + N(i) \right) - N(i), \begin{cases} I_n & k = k_0 \\ O & k > k_0 \end{cases}.
\]

If we consider an admissible initial state \(x(0)\) and a sequence control \(u(j), j = 0, 1, \ldots, k+q-1\), the state solution of the system (5) at time \(k\) is given by

\[
\bar{x}(k) = \begin{bmatrix}
\bar{x}_1(k) \\
\bar{x}_2(k)
\end{bmatrix} = \begin{bmatrix}
I_{n_1} \\
0
\end{bmatrix} \Phi_1(k, s) \bar{x}_1(s) - \sum_{j=s}^{k-1} \Phi_1(k, j + 1)E_1^{-1}(j)D_1(j)u(j) \\
+ \sum_{j=k}^{k+q-1} H(k, j) u(j), \ k \geq s,
\]

where \(H(k, j) = \Psi_2(k, j) (I_n + N(j))^{-1} D_2(j), j = k, \ldots, k+q-1\) and \(q\) is related to the nilpotence index of matrices \(\Psi_2(k, N).\)

Taking account the solution of the periodic system we have the following result.

**Proposition 1** Consider the \(N\)-periodic Leontief dynamic system \(\left( \bar{E}(\cdot), \bar{A}(\cdot), \bar{D}(\cdot) \right)_N\). The system has an admissible solution if and only if \(E_1^{-1}(k) + I_n \geq 0, -E_1^{-1}(k)D_1(k) \geq 0\) and \(H(k, j) \geq 0, j = k, \ldots, k+q-1,\) where \(H(k, j) = \Psi_2(k, j) (I_n + N(j))^{-1} D_2(j)\).

Normally in economic problems the initial stock must be positive then it is usual that

\[0 < \Phi_1(k, s)x_1(s)\]

and

\[\Phi_1(k, s)x_1(s) \geq \sum_{j=s}^{k-1} \Phi_1(k, j + 1)E_1^{-1}(j)D_1(j)u(j),\]

then it is sufficient that \(H(k, j) \geq 0, j = k, \ldots, k+q-1\) and choosing an initial condition satisfying condition (7).

We summarize the above results in the following proposition.
Proposition 2 Consider the $N$-periodic Leontief dynamic system $\left(\bar{E}(\cdot), \bar{A}(\cdot), \bar{D}(\cdot)\right)_N$. If
\[ \Psi_2(k,j)(I_{n_2} + N(j))^{-1} D_2(j) \geq 0 \]
and choosing an initial condition satisfying condition (7) then the system has an admissible solution.

3 Conclusions

In this work, the $N$-periodic Leontief dynamic model is considered. Some conditions to obtain admissible solutions to the $N$-periodic economic model are obtained. In particular, the nonnegativity of some matrices constructed from the initial coefficient matrices is sufficient to obtain a nonnegative trajectory of the discrete-time descriptor system.

References


