Synthesis of Stable Takagi-Sugeno Fuzzy Systems

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Abstract: - The paper deals with the problem of synthesis of a controller to the plant described by Takagi-Sugeno fuzzy model with linear input-output submodels in the consequents of rules. A procedure for design of the controller guaranteeing stability of closed-loop system based on polynomial approach is presented. The controller assigns the same characteristic polynomial for all feasible linear subsystems. The presented method can be used for both continuous-time and discrete-time systems.

Key-Words: - fuzzy system, stability, polynomial, Diophantine equation, input-output model, LTV system

1 Introduction
Takagi-Sugeno fuzzy systems with linear submodels in consequents of rules offer a good possibility to approximate dynamical behaviour of nonlinear systems. One of the advantages of such systems is that controllers for Takagi-Sugeno fuzzy systems can be designed by well-known methods used in theory of linear systems. The problem is stability analysis of such closed-loop systems. The main results concerning stability analysis of such systems were achieved by Wang, Tanaka and Griffin in [8]. This work concerns Takagi-Sugeno fuzzy systems with linear state-space submodels. They proved that the problem of stability analysis of such systems is equivalent to finding a common positive definite matrix to the quadratic Lyapunov equation and suggested some procedures how to find it. Some results were published also in [1], [4] and [6]. It was shown that the problem of stability analysis of Takagi-Sugeno fuzzy systems can be solved by testing robust stability of polynomials with polynomial structure of their coefficients.

Some results concerning the synthesis of controllers guaranteeing stability of closed-loop system can be found for example in [2] for Takagi-Sugeno fuzzy systems with state-space submodels and in [3] for Takagi-Sugeno fuzzy systems with input-output submodels. In both cases the Takagi-Sugeno fuzzy model of plant and Takagi-Sugeno fuzzy controller are supposed.

In this paper a new method for design of a linear time-variant controller to a plant described by Takagi-Sugeno fuzzy model with linear input-output submodels in the consequents of rules is presented. The proposed controller assigns the same characteristic polynomial for all feasible linear subsystems and guarantees stability of a closed-loop system. The presented results are valid under assumption of slowly varying parameters of the model or weights of rules. The presented method can be used for both continuous-time and discrete-time systems.

2 Takagi-Sugeno fuzzy model of the plant
The Takagi-Sugeno fuzzy model of the plant is considered, where the rules are written in the following form:

\[ R_i : \text{IF } v_i(t) \text{ is } M_i^f \text{ and } \cdots \text{ and } v_s(t) \text{ is } M_s^f \text{ THEN } y_i(t) = -\sum_{j=1}^{k_i} a_{i,j}^f y_j(t) + \sum_{j=0}^{k_i} b_{i,j}^f u_j(t) \] (1)

\[ R_j : \text{IF } v_i(t) \text{ is } M_i^p \text{ and } \cdots \text{ and } v_s(t) \text{ is } M_s^p \text{ THEN } y_i(t) = -\sum_{j=1}^{k_j} a_{i,j}^p y_j(t-j) + \sum_{j=0}^{k_j} b_{i,j}^p u_j(t-j) \] (2)

\[ i = 1,2,\ldots, R, \text{ for continuous-time case and for discrete-time case respectively, where} \]

\[ v(t) = [v_1(t), \ldots, v_s(t)] \] variables of the premise (some measurable plant variables)

\[ M_j^f \text{ fuzzy sets} \]

\[ y(t) \in \mathbb{R} \] the output of the plant

\[ u(t) \in \mathbb{R} \] the input to the plant
The following procedure will be derived for discrete-time case, but it can be followed for continuous-time system analogically. 

The total output of the fuzzy system is:

\[
y(t) = \sum_{i=1}^{R} w_i(v(t))y_i(t) = \sum_{i=1}^{R} h_i(v(t))y_i(t) = \sum_{i=1}^{R} h_i(v(t))\left(\sum_{j=-\infty}^{\infty} a_{ij}y(t-j) + \sum_{j=0}^{\infty} b_{ij}u(t-j)\right)
\]

where

\[
w_i(v(t)) = \prod_{j=1}^{M_i} M_j(v_j(t))
\]

\(M_j(v_j(t))\) is the grade of membership of \(v_j(t)\) in \(M_j\). It is assumed that

\[w_i(v(t)) \geq 0, \text{ for } i = 1, 2, \ldots, R \text{ and } \sum_{i=1}^{R} w_i(v(t)) > 0 \text{ for all } t.\]

Therefore

\[
\sum_{i=1}^{R} h_i(v(t)) = 1.
\]

By applying Z-transform to (3) under the assumption of slowly varying weights \(h_i\) the following transfer function of the model of plant is obtained:

\[
G_p(z, h) = \frac{Y(z, h)}{U(z, h)} = \frac{\sum_{i=1}^{R} w_i(v(t))G_p^i(z)}{\sum_{i=1}^{R} w_i(v(t))} = \sum_{i=1}^{R} h_i(v(t))G_p^i(z) = \sum_{i=1}^{R} h_i(v(t))\sum_{j=0}^{\infty} b_{ij}z^j = \sum_{i=1}^{R} h_i(v(t))\sum_{j=0}^{\infty} a_{ij}z^j = \frac{b(z, h)}{\sum_{i=1}^{R} h_i(v(t))d(z, h)}
\]

where \(G_p^i(z)\) denotes the transfer function of the consequent of the \(i\)-th rule.

The main idea of this paper consists in using polynomial methods for design of a controller to the Takagi-Sugeno fuzzy system described above which guarantees stability of closed-loop system. In order to assure stability of such systems, methods designated for linear time-invariant (LTI) systems are often used ([3],[6]). However, this is not possible generally. It has to be emphasized that stability of linear time-variant (LTV) systems using stability analysis of corresponding frozen-time systems can be guaranteed only under assumption of slowly varying parameters (see [7]) which is often omitted. The question how big the rate of their changing is allowed remains unanswered. Nevertheless, the experiments reveal that in most cases the designed controller stabilizes the plant (2).

Our aim is to find a controller that would assign the same characteristic polynomial to all LTI subsystems (for all admissible variations of weights \(h_i\)). Filev in [3] proved that if \(b(z)=b(z)\) for all \(i=1,\ldots,R\) where \(b(z)\) is a Schur polynomial then a stabilizing controller can be always found as a fuzzy parallel distributed compensator (PDC). It means that the controller can be described by Takagi-Sugeno fuzzy model with input-output submodels in the consequents of rules sharing the same antecedents with the model of the plant. It can be easily shown that this is not possible for the numerators \(b(z)\) being mutually different even by using a fuzzy non-PDC.

The following section deals with the synthesis of an LTV controller stabilizing plant (2) in a closed loop. Let us once more remind that the results are valid only under assumption of slowly varying weights \(h_i, i=1,\ldots,R\).

3 Design of the controller
The problem of finding a stabilizing controller to a plant described by the Takagi-Sugeno fuzzy model is solved assigning the same characteristic polynomial in closed-loop system for all feasible LTI subsystems.

Assume that the plant is described by Takagi-Sugeno fuzzy model (2) and the weights \(h_i, i=1,\ldots,R\) are slowly varying. Let the controller be an LTV system expressed by its transfer function

\[
G_c(z, h) = \frac{r(z, h)}{q(z, h)}
\]

where
The coefficients \( r, q \) are some functions of the vector of weights of rules \( h \). Our task is to find these functions so that the closed-loop system is stable.

As it was mentioned above our goal is to assign the same characteristic polynomial of the closed-loop system for all \( h \in \mathbf{H} \), i.e.:

\[
a(z, h)g(z, h) + b(z, h)r(z, h) = \Delta(z)
\]

where \( \Delta(z) \) is a Schur polynomial of appropriate degree.

The well-known condition of solvability of (9) is that the greatest common divisor of polynomials \( a(z, h) \) and \( b(z, h) \) divides the polynomial \( \Delta(z) \) for all \( h \in \mathbf{H} \), i.e.

\[
g((z, h), b(z, h))\Delta(z).
\]

In order to check whether the condition (10) is satisfied one can use the well-known algorithm for finding the greatest common divisor of two polynomials [5] with respect to the vector variable \( h \). However the better solution consists in solving the Diophantine equation (9). If the solution does not exist for some \( h \in \mathbf{H} \) it means that for those \( h \in \mathbf{H} \) the condition (10) is violated. The greatest common divisor can be then easily determined substituting the vector \( h \) in (6) and using the standard algorithm mentioned above. If \( g(a, b) \) is Schur polynomial then a new polynomial \( \Delta(z) \) has to be chosen to be able to solve equation (9) (for example \( \Delta(z) = \Delta(z)^n g \) always leads to a solvable equation (9)). If \( g(a, b) \) is not Schur polynomial the controller (7) stabilizing the plant (2) by assigning the same characteristic polynomial to all linear subsystems can not be found.

As in the closed-loop system at least one step delay has to be guaranteed, \( \deg r(z, h) < \deg q(z, h) \). The degrees of polynomials \( r(z, h), q(z, h) \) and \( \Delta(z) \) can be found comparing degrees of polynomials of both sides of equation (9):

\[
\deg \Delta > 2 \deg a - 1
\]
\[
\deg r < \deg a - 1
\]
\[
\deg q = \deg \Delta - \deg a.
\]

Substituting (6) and (8) to (9) the following equation is obtained:

\[
\sum_{j=0}^n h_j \sum_{j=p}^n d_j \cdot \sum_{j=p}^n q_j z^j + \sum_{k=0}^m h_k b_k z^k \cdot \sum_{j=p}^n r_j z^j = \Delta(z)
\]

All solutions of (12) can be expressed as

\[
r(z, h) = r_0(z, h) + f(z)\Delta(z) \]
\[
q(z, h) = q_0(z, h) - f(z)\Delta(z)
\]

where \( r_0(z, h) \) and \( q_0(z, h) \) are the particular solutions of (12) and \( f(z) \) is an arbitrary polynomial.

The determination of polynomials \( r_0(z, h) \) and \( q_0(z, h) \) is equivalent to solving the system of \( (\deg \Delta + 1) \) linear equations arising from (12) by comparing of all coefficients in \( z^i \), \( i = 0, \ldots, \deg \Delta \). As the solutions \( r, q \) of this system are functions of \( h \) the general solution is relatively complicated. To better understand the described procedure its principle will be shown on two illustrative examples.

### 3.1 Example 1

For sake of simplicity assume a plant described by the following two-rule Takagi-Sugeno fuzzy model:

IF \( y(t-1) \) is \( \mathcal{M}^1 \)
  \[ y(t) = -0.1y(t-1) + u(t) \]
IF \( y(t-1) \) is \( \mathcal{M}^2 \)
  \[ y(t) = -0.3y(t-1) + u(t) + 0.4u(t-1) \]

Fig. 1 Membership functions

The transfer function of the model is:

\[
G_p(z, h) = \frac{b(z, h)}{a(z, h)} = \frac{h_1 b^1(z) + h_2 b^2(z)}{h_1 a^1(z) + h_2 a^2(z)}
\]

\[
= \frac{h_1 b^1(z) + (1-h_2) b^2(z)}{h_1 a^1(z) + (1-h_2) a^2(z)}
\]

where
From (11) the degrees of polynomials Δ(z), r(z,h) and q(z,h) can be determined as follows:

\[ \deg \Delta = 2, \quad \deg r = 0, \quad \deg q = 1. \]

Choose the global characteristic polynomial Δ(z) as

\[ \Delta(z) = (z + 0.1)^2. \]

By solving the Diophantine equation (9) in the form

\[
\begin{align*}
[h_1(z + 0.1) + (1 - h_1)(z + 0.3)]q_1(h_1)z + q_0(h_1) + \\
+h_1z + (1 - h_1)(z + 0.4)]r_0(h_1) = (z + 0.1)^2
\end{align*}
\]

one can obtain

\[ q_1(h_1) = 1, \]
\[ q_0(h_1) = \frac{0.08h_1^2 - 0.12h_1 + 0.05}{0.2h_1 - 1}, \]
\[ r_0(h_1) = \frac{-0.04h_1^2 + 0.08h_1 - 0.04}{0.2h_1 - 1}. \]

It is not difficult to see that for \( h_1 = 0.5 \) there is no solution of the Diophantine equation. Really, if \( h_1 = 0.5 \) then the greatest common divisor

\[ g(a(z,0.5),b(z,0.5)) = g(z + 0.2, z + 0.2) = z + 0.2 \]

does not divide the polynomial \( \Delta(z) \) and the necessary condition of solvability of (10) is violated.

Choose the polynomial \( \Delta(z) \) as

\[ \Delta(z) = (z + 0.1)(z + 0.2). \]

One particular solution of (9) can be found as

\[ q_1(h_1) = 1, \]
\[ q_0(h_1) = 0.4h_1 - 0.2, \]
\[ r_0(h_1) = -0.2h_1 + 0.2 \]

that corresponds to the LTV controller with transfer function

\[ G_c(z,h_1) = \frac{-0.2h_1 + 0.2}{z + 0.4h_1 - 0.2} \]

or equivalently in the time-domain

\[ u(k) = (0.2 - 0.4h_1)u(k - 1) + (0.2 - 0.2h_1)e(k - 1). \]

In Fig. 2 and Fig. 3 the responses on initial condition of the closed-loop system are shown for \( c = 0.1 \) and \( c = 1 \) respectively. It is evident that the controller stabilizes the plant for arbitrary rate of changing of \( h_1 \).

### Fig. 2 Response of closed-loop system for \( c = 0.1 \)

### Fig. 3 Response of closed-loop system for \( c = 1 \)

#### 3.1 Example 2

Assume a plant described by the following two-rule Takagi-Sugeno fuzzy model composed from unstable submodels:
IF $y(k-1)$ is THEN $y(k)=-2y(k-1)+u(k)$
IF $y(k-1)$ is THEN $y(k)=-3y(k-1)+u(k)+u(k-1)$

Choose the global characteristic polynomial $\Delta(z)$ as

$$\Delta(z) = (z + 0.5)^2.$$  

The Diophantine equation (9) turns to

$$[h_1(z + 2) + (1 - h_1)(z + 3)]q_1(h_1)z + q_0(h_1)] + [h_1z + (1 - h_1)(z + 1)]r_0(h_1) = (z + 0.5)^2$$  

from which one solution can be obtained as

$$q_1(h_1) = 1$$  
$$q_0(h_1) = 0.5h_1^2 - 1.5h_1 + 1.125$$  
$$r_0(h_1) = -0.5h_1^2 + 2.5h_1 - 3.125.$$  

The transfer function of the controller is

$$G_c(z, h_1) = \frac{-0.5h_1^2 + 2.5h_1 - 3.125}{z + 0.5h_1^2 - 1.5h_1 - 1.125}$$  

or equivalently in the time-domain

$$u(k) = (0.5h_1^2 - 1.5h_1 - 1.125)u(k-1) + (-0.5h_1^2 + 2.5h_1 - 3.125)u(k-1).$$

In Fig. 5 and Fig. 6 the responses on initial condition of the closed-loop system are shown for $c=10$ and $c=1$ respectively.

The figures show that the controller stabilizes the closed loop only for slowly varying weight of rule $h_1$ ($c=10$). If the rate of variance is higher ($c=1$) then the proposed method does not guarantee the stability.

**4 Conclusion**

The new method for finding a linear time-variant controller to a plant described by Takagi-Sugeno fuzzy model with linear input-output submodels in the consequents of rules guaranteeing stability of the closed-loop system was introduced. The method is based on assigning the same characteristic polynomial for all feasible subsystems. The proposed method guarantees the stability only for slowly varying vector of weights of rules $h$.

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