PREFERENCE RANKING AND DECISIONS BASED ON FUZZY EXPERT INFORMATION.
ALEXEY L. SADOVSKI
Department of Computing and Mathematical Sciences
Texas A&M University-Corpus Christi
6300 Ocean Drive, Corpus Christi, TX 78412, USA

Abstract: This paper deals with group decisions based on the rating methods of fuzzy and regular preference rankings. The preference ranking is one of the methods to solve so-called selection problems. Selection problems are very important for decision making in unique systems such as medical, environmental or ecological ones. Very often the right decision is based upon expert information. In this paper we deal with some approaches to the choice of the best variants [1], and its application to the multi-criteria optimization and decisions. This paper presents axiomatic systems of rating methods of preference ranking based on fuzzy expert information. Results include the convergence of consensus ranking to the real ranking almost everywhere, and the inclusion of the consensus ranking into the Kemeny Median [2]. It is shown, that contemporary rating systems, (for instance those used in sports classifications, [3]), are congruent in the sense of producing the same final preference ranking.

Key-Words: group decisions, preference ranking, consensus, fuzzy expert information

1. Preference Ranking.
Let us consider a finite set of objects \( A = \{a_1, \ldots, a_n\} \) and a set of \( m \) experts \( E \). Each expert presents binary matrix \( Q_k = (q_{ij}) \), \( k=1, \ldots, m \), and \( ij=1, \ldots n \). The problem is to find consensus ranking of objects based upon information provided by experts.

There are a few ways to solve the problem under consideration. The first one is to determine Pareto set \( P = \{P \cap Q_k \subseteq P \cap Q_l \} \), but set \( P \) is too wide. The second method of the solution presented by Arrow [4] was based on a contradictory system of five axioms. The most useful result obtained by Kemeny, is the so-called Kemeny Median: \( H = \{K \mid \sum d(K, Q_k) = \min \sum d(P, Q_k)\} \), which can be determined by methods of integer programming. It is necessary to outline, that the Kemeny Median satisfies four of five Arrow’s axioms. Also, there is an inclusion \( H \subseteq P \), but still the set \( H \) is quite wide.

The methods presented in this paper have three advantages. First, there is a very simple numerical procedure. The second improvement is the possibility of using different forms of expert information such as preference ranking, binary comparison, and multi-comparison at the same time; this significantly differs from previous methods based on uniform types of expert information. The third advantage is that the result of this rating procedure is a unique preference ranking and not just some set of suitable alternatives such as a Pareto Set or a Kemeny Median.

2. Axioms of Rating Systems
Suppose there is some (maybe unknown to the decision-maker) order of objects \( a_1, \ldots, a_n \) under consideration. Let us assume, that we have chosen some arbitrary scale, and each object has its own yet unknown value \( r(a_i) \) in this chosen scale. The following is an axiom of existence:
**Axiom 1.** There is some order of given objects in any chosen scale.

Let us denote by \( \Delta_k = r_i - r_j \) the difference between real rating values. We believe that binary relationships given by experts satisfy

\[
\text{Axiom 2: } \frac{\text{number of preferences } a_i > a_j}{\text{number of preferences } a_j > a_i} = f(\Delta_k),
\]

where \( f(\Delta) \) is a nonnegative and strictly increasing function such that \( f(0) = 1 \). This assumption shows the odds or fuzzy odds of preferences by experts, who are asked to rank or compare given objects. Let \( r_{k0}, i = 1, ..., n \) to be some arbitrary initial ratings, \( r_{ik} \) is rating of an \( i \)-th object after \( k \)-th recalculation, and \( \Delta_k = r_i(k) - r_j(k) \).

The following statement gives the simple way to calculate ratings of objects according to the results of expert estimations.

**Axiom 3.**

\[
\begin{align*}
q_i(k) &= \begin{cases} 
1, & \text{if } a_i > a_j \\
0, & \text{if } a_i = a_j \\
-1, & \text{if } a_i < a_j 
\end{cases} \\
r_i(k) &= r_i(k-1) + q_i(k) F(\Delta_i(k-1)) \\
r_j(k) &= r_j(k-1) + q_j(k) F(\Delta_j(k-1))
\end{align*}
\]

In the case of fuzzy information coefficients \( q \) are equal values of membership function of a fuzzy binary relationships provided by experts respectively. For fuzzy relationships formulas in axiom 3 should be change as follows:

\[
\begin{align*}
r_i(k) &= r_i(k-1) + q_i(k) F(\Delta_i(k-1)) \\
-q_j(k) F(\Delta_j(k-1)) \\
r_j(k) &= r_j(k-1) + q_j(k) F(\Delta_j(k-1)) \\
-q_i(k) F(\Delta_i(k-1))
\end{align*}
\]

which means an increase proportional to fuzzy preference and a decrease proportional to fuzzy non-preference.

It is reasonable that for a large \( \Delta_k \) increasing of \( r_k \) should be small, if \( a_i > a_j \), but decreasing should be large if \( a_i < a_j \). This idea is very useful, for instance, in some sports methods of classifications: it means that if a strong team or player outs the weak one then there is almost no increasing in the rating for the winner. However in the case of losing the game the higher rated team should loose many points. That is why we have two following assumptions for function \( F \):

**Axiom 4.**

\[
\lim_{\Delta \to \infty} F(\Delta) = 0
\]

**Axiom 5.**

\[
\lim_{\Delta \to -\infty} F(\Delta) = L > 0
\]

**Example:** The best known rating system has been suggested by Elo [3], and it is used for the ranking of chess players. Elo supposed that 200 points are difference between neighbor grades of players, and that probability of winning of the more qualified player is equal to 0.75. So he had chosen some scale and calculated ratings for all more or less well known players since the beginning of the century. Using the simplest function satisfied axiom 2, which is an exponent, we can determine base of this function:

\[
\frac{a^{200}}{a^{200} + 1} = 0.75 \quad \text{and } a = 1.0055
\]

The new rating in Elo system is equal to old rating plus ten folded difference between an actual result of the match and expected outcome of the game.

### 3. Some Results.

The following theorem establishes equivalency for all rating systems of preference ranking including rankings based on fuzzy information.

**Theorem 1.** For any initial ratings any method based on axioms 1 through 5 presents some preference ranking which is the same as a real unknown ranking with
probability one in the space of realization, when \( k \to \infty \).

A sketch of the proof is the following. Let us consider ranking of the two objects \( a_i \) and \( a_j \) as a rating procedure. Hence we deal with the space of sequences \( \Omega = (\omega_1, \omega_2, \ldots) \), where \( \omega_k = 1 \) if \( a_i \succ a_j \) and \( \omega_k = -1 \) otherwise. Each \( \omega \) is realization of Bernoulli trials and there is some measure \( \mu \) in the space \( \Omega \) which is generated by probability

\[
p = \frac{f(\Delta_{ij})}{1 + f(\Delta_{ij})}
\]

Suppose that \( \Delta_{ij} > 0 \), i.e. \( p \succ \frac{1}{2} \). On the one hand according the weak law of large numbers almost everywhere

\[
\lim_{k \to \infty} [E(\Delta_{ij}(k)) - E(\Delta_{ij}(k - 1))] = 0
\]

On the other hand

\[
\Delta_{ij}(k) = \Delta_{ij}(k - 1) + 2q_{ij}F(\Delta_{ij}(k - 1)).
\]

So with probability one in the space of realization

\[
E[\Delta_{ij}] = E[\Delta_{ij} + 2q_{ij}F(\Delta_{ij})],
\]

where \( q_{ij} \) is a random variable. In such a way we can obtain the following equation with respect to \( \Delta_{ij} \):

\[
pF(\Delta_{ij}) - (1 - p)F(-\Delta_{ij}) = 0.
\]

Because of continuity and decreasing of function \( F \) and because of \( p \succ \frac{1}{2} \) there is some positive \( \Delta_{ij} \) which is solution of last equation. So, if \( \Delta_{ij} > 0 \), then as a result of rating procedure \( \Delta_{ij}(k) > 0 \), if \( k \to \infty \) with probability one.

Now we can consider comparison of many objects. Here we deal with matrix

\[
B_k = (\Delta_{ij}(k))_{i,j=1,\ldots,n}\text{ and with more complicated ergodic process [6]. The measure in this case is constructed as the measure of Cartesian product of one dimensional sequences. And if for some } a_i \text{ and } a_j \text{ then } \Delta_{ij} > 0 \text{ then } a_i \succ a_j.
\]

4. Ratings and Kemeny Median.

Presume that experts present information about preferences in matrix form: \( Q_k = (q_{ij}(k)), i,j = 1, \ldots, n, k = 1, \ldots, m \). For any two matrices of binary relations \( Q_k \) and \( Q_l \) distance between them may be defined in the following way:

\[
d(Q_k, Q_l) = \frac{1}{2} \sum_{i,j} |q_{ij}(k) - q_{ij}(l)|
\]

If matrices \( Q_k \), \( Q_l \) present preference ranking by experts then Kemeny median \( (4) \) is such matrix \( K \) that

\[
\sum_{i,j=1}^{m} d(K, Q_k) = \min \sum_{i,j=1}^{m} d(P, Q_k)
\]

The Kemeny median is really a set of such matrices \( K \), and at a present moment it is considered the most useful consensus ranking, however the determination of this median is the problem of integer programming with all difficulties of calculations. The following result establishes connection between Kemeny median and rating rankings:

**Theorem 2.** The consensus preference ranking obtained as a result of a rating procedure belongs with probability one to the Kemeny Median set in the space of all realizations \( \Omega \).

It is easily seen that rating methods could be used as iterative procedures for determining Kemeny Median. The proof of this theorem is based on facts that Kemeny Median as well as a rating preference ranking are satisfied four of five Arrow axioms.

**Remarks.** This theorem shows that all rating systems are equal in the sense that with probability one they would produce the same consensus ranking. Indeed, we have just discussed so called additive rating systems of preference ranking, but the same statements and ideas are correct for multiplicative rating methods in which

\[
r_t(k) = r_t(k - 1) \cdot q_{ij}(k) \cdot F(\Delta_{ij}(k - 1)).
\]
5. Applications to Multi-objective Optimization.

The problem consists of finding such $x$ from the set of feasible decisions $X$ that gives, in some sense, the best value to objective functions $v_1(x), \ldots, v_n(x)$. If the set $X$ is a finite set, then we have a selection problem of decision-making. If $X$ belongs to some continuous space, then we deal with a problem of multi-objective programming.

There are many approaches to the solution of such problems under consideration. The most effective approaches to the solution are based on determining some utility function $G(v_1(x), \ldots, v_n(x))$ of given criteria. The next step is to solve problem of the mathematical programming:

$$\max \rightarrow G(v_1(x), \ldots, v_n(x)), \text{where } x \in X$$

Usually function $G(\cdot)$ depends on preferences of decision-makers or it is based upon expert information. Let us consider some linear-weighted aggregate objective function as an integrated objective function (utility function):

$$G = \alpha_1 v_1(x) + \ldots + \alpha_n v_n(x).$$

The question now is how to determine the weight coefficients. Using rating methods of preference ranking, we can ask decision-makers (expert, advisers etc.) to present their preferences of objective functions $v_1(x), \ldots, v_n(x)$ in the form of ranking, or binary, or multi-comparisons, and find coefficients $\alpha_k$ based on final consensus ranking [1, 7].

As a result of just described procedure of preference ranking we can obtain ratings $r_1, \ldots, r_n$ for given objective functions. Suppose that we have got $r_1 \geq r_2 \geq \ldots \geq r_n$. Let the value of coefficient $\alpha_i$ to be equal to one, in this case using the structure of the rating procedure we can find other weight coefficients from the following relationship:

$$\frac{\alpha_i}{\alpha_j} = f(\Delta_i), j = 1, \ldots, n,$$

where $f$ and $\Delta_i$ are defined by just described system of axioms of preference ranking. Respectively, the determined in such a way weight coefficients would give us an opportunity to use additive integrated utility function when we have to deal with few objectives at the same time.

$$G = \alpha_1 v_1(x) + \ldots + \alpha_n v_n(x).$$

Now we are able to transform an optimization problem with many objectives to the problem of mathematical programming with only one objective function, and apply one of the developed algorithms to determine optimal solution.

In the conclusion we would like to stress out that rating systems of preference ranking are very flexible. They provide an opportunity to work with different types of expert information such as binary and multi-comparison, ranking, etc. Moreover, there is a possibility to work with fuzzy information. If, for instance, $\mu_{ij}$ is measure of belonging that $a_i \succ a_j$, then it is enough to replace $q_{ij}$ by $\mu_{ij}$ in axiom 3 to use fuzzy relationship offered by experts. The last remark concerns theorem 1, which holds also under conditions of fuzziness.

References:


