CONSTRUCTION OF AN α-LABELING OF THE GRAPH $C_{4a} \cup C_{4b} \cup C_{4c}$

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Abstract: An α-labeling of a graph G is a graceful labeling of graph G with an additional property. In this paper the existence and construction of an α-labeling of the graph $C_{4a} \cup C_{4b} \cup C_{4c}$ is discussed.

Key-Words: Graph Labeling, α-labeling, Graceful Valuation

1. Introduction

A *graceful labeling* (or *β-labeling*) of a simple graph G = (V, E) with m = |V| vertices and n = |E| edges is a one-to-one mapping Ψ of the vertex set V(G) into the set {0, 1, 2, … , n} with this property: If we define, for any edge e ={u, v}∈ E(G) , the value $\Psi^{\bullet}(e) = |\Psi(u)-\Psi(v)|$ then $\Psi^{\bullet}$ is a one-to-one mapping of the set E(G) onto the set {1, 2, … , n}.
A graph is called graceful if it has a graceful labeling. An *α-labeling* (or *α-labeling*) of a graph G = (V, E) is a graceful labeling of G which satisfies the following additional condition: There exists a number γ ($0 \le \gamma \le |E(G)|$) such that, for any edge e ∈ E(G) with end vertices u, v ∈ V(G), min [Ψ(u), the Ψ(v)] ≤ γ < max [Ψ(u), Ψ(v)].

The concept of a graceful labeling and of an α-labeling were introduced by Rosa [4]. Rosa proved that if G has an α-labeling and if all vertices of G are of even degrees, then $|E(G)| \equiv 0 \pmod 4$ (and G is bipartite). In [4] it is also shown that these conditions are also sufficient if G is a cycle. Abrham and Kotzig [1] proved that the graph $C_{4a} \cup C_{4b}$ has an α-labeling for all positive integers a and b. Then the construction of an α-labeling of the graph $C_{4a} \cup 2C_{4b}$ for all positive integers a and b, except a = b =1, has been shown in [3] by Eshghi and Carter. This paper is closely related to their results.

One of the results of Abrham and Kotzig should be mentioned here: If G is a 2-regular graph on n vertices and n edges which has a graceful labeling Ψ then there exists exactly one number x ($0 < x < n$) such that $\Psi(v) \neq x$ for all v ∈ V(G); this number x is referred to as the missing value of the graceful graph [2].
The definitions of transformations type 1 and 2 and 3 can be found in [3]. These types of transformations are very useful for our considerations.
All parameters in this paper are positive integers.

2. Existence of α-labelings of Special Classes of Graphs

Proposition 1: The graph $C_{4(p-\Delta)} \cup C_{4p} \cup C_{4(m+\Delta)}$ has an α-labeling if $p \le m < 2p$ and $\Delta < (1/2)p - (1/2)$.

Proof: First consider the construction of an α-labeling of the graph $2C_{4p} \cup C_{4m}$ for $p \le m < 2p$ in [3]. Now remove all of the edges of the first C_{4p} in this construction. Furthermore, remove the edges (2p+1, 6p+4m-1), (p, 5p+4m) and (p, 5p+4m-1).

Then connect the two vertices 5p+4m-1 and 5p+4m to the vertex p+1, this yields the edge labels 4p+4m-2 and 4p+4m-1. The edge labeled 4p+4m is also generated by joining the two vertices 2p+1 and 6p+4m+1. Next we will form the following snake in such a way that its vertices are labeled consecutively as follows: (2p+1, 6p+4m+1, 2p, 6p+4m+2, ... , 2p-Δ+4, 6p+4m+Δ-2, 2p-Δ+3, 6p+4m+Δ-1, 2p-Δ+2, 6p+4m+Δ, 2p-Δ, 6p+4m+Δ+1, ... , 2p-2Δ+2, 6p+4m+2Δ-1, 2p-2Δ+1, 6p+4m+2Δ, 2p-2Δ). The values of the edges are then 4p+4m, 4p+4m+1, 4p+4m+2, ... , 4p+4m+2Δ-6, 4p+4m+2Δ-5, 4p+4m+2Δ-4, 4p+4m+2Δ-3, 4p+4m+2Δ-2, 4p+4m+2Δ, 4p+4m+2Δ+1, ... ,4p+4m+4Δ-3, 4p+4m+4Δ-2, 4p+4m+4Δ-1, 4p+4m+4Δ. The edge labeled 4p+4m+2Δ-1 is obtained by connecting the two vertices 2p-2Δ and 6p+4m-1. The above snake and the edge (2p-2Δ, 6p+4m-1) will transform the cycle C_{4m} to the cycle $C_{4(m+\Delta)}$.

Now the edge labeled 4p+4m+4Δ+1 are generated by joining the two vertices 2p-Δ+1 and 6p+4m+3Δ+2. Now we will have to distinguish the two following cases according to whether 2Δ+1 < p < 3Δ+3 or p ≤ 3Δ+3:

Case 1) 2Δ+1 < p < 3Δ+3

First we will describe the construction of another snake. In this case the consecutive vertices will be labeled as follows: (6p+4m+2Δ+1, 2p-2Δ-1, 6p+4m+2Δ+2, 2p-2Δ-2, ... , 7p+4m-2, p+2, 7p+4m-1, p, 7p+4m, ... , 6p+4m+3Δ, 2p-3Δ-1, 6p+4m +3Δ+1, 2p-3Δ-2, 6p+4m+3Δ+2). The values of the edges are then 4p+4m+4Δ+2, 4p+4m+4Δ+3, 4p+4m+4Δ+ 4, ... , 6p+4m-4, 6p+4m-3,6p+4m-1, 6p+4m, ... ,4p+4m +6Δ+1,4p+4m+6Δ+2,4p+4m+6Δ+3,4p+ 4m+6Δ+4.

The edges values 6p+4m-2 and 4p+4m+6Δ+5 are obtained by joining the following pairs of vertices respectively: 2p-Δ+1 and 8p+4m-1; 2p-4Δ-4 and 6p+4m+2Δ+1. Next the transformation type 1 is applied to the rest of vertex labels by choosing the two vertices 2p-4Δ-4 and 8p+4m-Δ-1 as end vertices. Thus the rest of the edge labels of $C_{4(p-\Delta)}$ will be generated.

Case 2) 3Δ+3 ≤p

We will have to distinguish the following cases:

2.1) 3Δ+3 ≤ p < 4Δ+4

First we apply transformation type 3 to the vertex labels (7p+4m-1, 7p+4m-2, ... , 6p+4m+3Δ+2, ... , 7p+4m-Δ-2, ... , 6p+4m+2Δ+2, 6p+4m+2Δ+1) and (2p-2Δ-1, 2p-2Δ-2, ... , p+4, p+3, p+2) by selecting the two vertices 6p+4m+3Δ+2 and 7p+4m-Δ-2 as end vertices. This transformation yields the edge labels 4p+4m+4Δ+2, 4p+4m+4Δ+3, 4p+ 4m+4Δ+4, ... , 6p+4m-4, 6p+4m-3. Now the two edge values 6p+4m-2 and 6p+4m-1 are generated by joining the following pairs of vertices respectively: 2p-Δ+1 and 8p-Δ+4m-1; p-Δ-1 and 7p+4m-Δ-2. Then we will use transformation type 1 to the vertex labels (0, 1, 2, ... , p-Δ-1, ... , p-1, p) and (7p+4m, 7p+4m+1, 7p+4m+2, ... , 8p+4m-Δ-1, ... , 8p+4m-1, 8p+4m) by considering the two vertices p-Δ-1 and 8p+4m-Δ-1 as end vertices. Therefore the cycle $C_{4(p-\Delta)}$ will be completed. In figure 1 the cycle $C_{4(p-\Delta)}$ is shown:

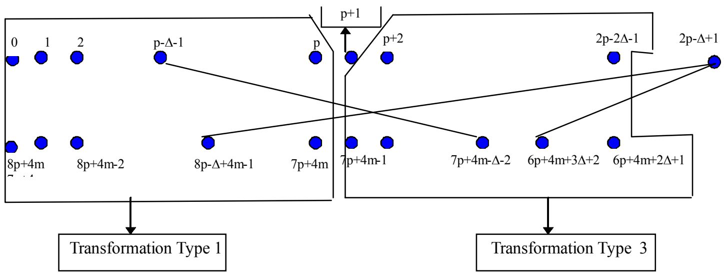

Figure 1: Cycle $C_{4p-4\Delta}$ in case 2.1 in proposition 1

2.2.) p = 4Δ+4

The first transformation in this case will be transformation type 1 to the vertex labels (p+2, p+3, ... , p+Δ+3, ... , 2p-2Δ-2, 2p-2Δ-1) and (6p+4m+2Δ+1, 6p+4m+2Δ+2, ... , 6p+4m+2+3Δ, ... ,7p+4m-4, 7p+4m-3, 7p+4m-2) where the two vertices p+Δ+3 and 6p+3Δ+2 are end vertices. Thus the edge labels 4p+4m+4Δ+2, 4p+ 4m+ 4Δ+3, ... , 6p+4m-5, 6p+4m-4 will be generated by this transformation. Then the two edge labels 6p+4m-3 and 6p+4m-2 are obtained by joining the following pairs of vertices respectively: p+Δ+3 and 7p+4m+Δ; 2p-Δ+1 and 8p+4m-Δ-1. Next we apply the transformation type 3 to the rest of vertex labels by choosing the vertex labels $8p+4m-\Delta-1$ and $7p+4m+\Delta$ as end vertices

2.3) $4\Delta+4 < p$

This case will be the same as case 2.1 by considering this fact that the position of the two vertices $6p+4m+3\Delta+2$ and $7p+4m-\Delta-2$ must be replaced in vertex sequence.

Lemma 1: The graph $C_{12c-8} \cup C_{8c-4} \cup C_{4c}$ has an α-labeling.

Proof: The proof of this lemma can be easily obtained from an α-labeling of the graph $3C_{8c-4}$. The reader is referred to [3].

Corollary 1: The graph $C_{4a} \cup C_{4b} \cup C_{4c}$ has an α-labeling if $2b \le a+c < 3b$, $a > b > c$ and $a < b+c$.

Proof: In proposition 1 assume $a = m+\Delta$, $b = p$ and $c = p-\Delta$. Thus $p = b$, $\Delta = b-c$ and $m = a-b+c$.

Proposition 2: The graph $C_{4(p+\Delta)} \cup C_{4(m-\Delta)} \cup C_{4p}$ has an α-labeling if $(1/2)\, p < m \le p-1$ and $\Delta < (1/2)m$.

Proof: For $(1/2)\, p < m \le p-1$ the construction of $2C_{4p} \cup C_{4m}$ have been discussed in [3]. Now remove all of the edges of C_{4m} and also the edges $(4p-2, 4p+4m+1)$, $(4p+m, 4p+5m+1)$, $(4p+m, 4p+5m+2)$ in this construction. Then we will connect the two vertices $4p+5m+1$ and $4p+5m+2$ to the vertex labeled $4p+m-1$ to yield the edge labels $4m+2$ and $4m+3$. The edge labeled $4m-4\Delta$ is generated by connecting the two vertices $4p+\Delta-1$ and $4p+4m-3\Delta-1$. Next we will assume that $m \ne 2\Delta+1$ and form the following snake in such a way that its vertices are labeled consecutively as: $(4p+4m+1, 4p, 4p+4m, 4p+1, \dots, 4p+4m-\Delta+3, 4p+\Delta-2, 4p+4m-\Delta+2, 4p+\Delta, 4p+4m+1-\Delta, 4p+\Delta+1, 4p+4m-\Delta, \dots, 4p+2\Delta-1, 4p+4m-2\Delta+2, 4p+2\Delta, 4p+4m-2\Delta+1, 4p-2)$. The values of the edges are then $4m+1, 4m, 4m-1, \dots, 4m-2\Delta+5, 4m-2\Delta+4, 4m-2\Delta+2, 4m-2\Delta+1, 4m-2\Delta, 4m-2\Delta-1, \dots, 4m-4\Delta+3, 4m-4\Delta+2, 4m-4\Delta+1, 4m-2\Delta+3$. In fact the above snake is joined to the second C_{4p} and convert that to the cycle $C_{4p+4\Delta}$.

Now we will distinguish the following cases according to whether $3\Delta+2 < m$, $2\Delta+2 < m \le 3\Delta+2$, $m = 2\Delta+2$ or $m = 2\Delta+1$:

Case 1) $3\Delta+2 < m$

This case will have three minor cases as follows.

1.1) $3\Delta+2 < m < 4\Delta+4$

First we apply transformation type 3 to the vertex labels $(4p+4m-2\Delta, 4p+4m-2\Delta-1, \dots, 4p+3m+\Delta+3, \dots, 4p+4m-3\Delta-1, \dots, 4p+3m+3, 4p+3m+2)$ and $(4p+m-2, 4p+m-3, \dots, 4p+2\Delta+2, 4p+2\Delta+1)$ by considering the two vertices $4p+3m+\Delta+3$ and $4p+4m-3\Delta-1$ as end points. Therefore the edge labels $2m+4, 2m+5, 2m+6, \dots, 4m-4\Delta-2, 4m-4\Delta-1$ are generated by this transformation. The two edge labels $2m+2$ and $2m+3$ are generated by joining the following pairs of vertices respectively: $4p+\Delta-1$ and $4p+2m+\Delta+1$; $4p+m+\Delta$ and $4p+3m+\Delta+3$. The rest of the edge labels will be generated if we use transformation type 1 to the rest of vertex labels if two vertices $4p+m+\Delta$ and $4p+2m+\Delta+1$ select as end vertices. This transformation completes the construction of cycle $C_{4m-4\Delta}$. The cycle $C_{4m-4\Delta}$ is shown in figure 2:

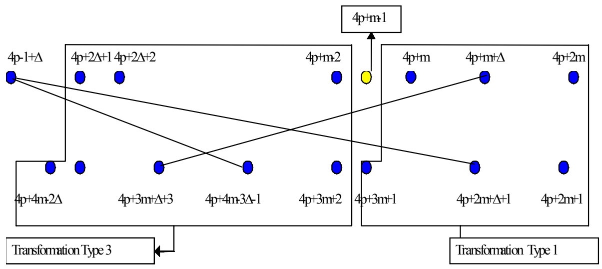

Figure 2: Cycle $C_{4m-4\Delta}$ in case 1.1 of proposition 2

1.2) $m = 4\Delta+4$

First we perform transformation type 1 to the vertex labels $(4p+2\Delta+1, 4p+2\Delta+2, \dots, 4p+m-\Delta-3, \dots, 4p+m-3, 4p+m-2)$ and $(4p+3m+3, 4p+3m+4, \dots, 4p+4m-3\Delta-1, \dots, 4p+4m-2\Delta-1, 4p+4m-2\Delta)$ by using the vertices $4p+m-\Delta-3$ and $4p+4m-3\Delta-1$ as end vertices. This transformation yields the edge labels $2m+5, 2m+6, \dots, 4m-4\Delta-2, 4m-4\Delta-1$. Then the two edge labels $2m+3$ and $2m+4$ are obtained by joining the following pairs of vertices respectively: 4p+Δ-1 and 4p+2m+Δ+2; 4p+m-Δ-3 and 4p+3m-Δ+1. Finally in order to generate the rest of the edge labels we will have to use transformation type 3 to the rest of vertex labels by considering the two vertices 4p+3m-Δ+1 and 4p+2m+Δ+2 as end points. This transformation will complete the construction of cycle $C_{4m-4\Delta}$.

1.3) 4Δ+4 < m

This case is the same as case 1.1 with the exception of changing the position of the vertices labeled 4p+4m-3Δ-1 and 4p+3m+Δ+3 to each other in each sequence.

Case 2) 2Δ+2 < m ≤ 3Δ+2

We will form the following snake in such a way that its vertices have the successive values (4p+4m-2Δ, 4p+2Δ+1, 4p+4m-2Δ-1, 4p+2Δ+2, ... , 4p+m-3, 4p+3m+3, 4p+m-2, 4p+3m +2, 4p +m, 4p+3m+1, ... , 4p+3Δ+1, 4p+4m-3Δ, 4p+3Δ+2, 4p+4m-3Δ-1). The edge labels of this snake are then 4m-4Δ-1, 4m-4Δ-2, 4m-4Δ-3, ... , 2m+6, 2m+5, 2m+4, 2m+2, 2m+1, ... , 4m-6Δ-1, 4m-6Δ-2, 4m-6Δ-3. The two edge labels 2m+3 and 4m-6Δ-4 are obtained by joining the following pairs of vertices respectively: 4p-1+Δ and 4p+2m+Δ+2; 4p+4Δ+4 and 4p+4m-2Δ. Then we apply transformation type 1 to the rest of vertex labels by considering the two vertices 4p+4Δ+4 and 4p+2m+2+Δ as end vertices. Therefore the rest of the edge values will be generated and the construction of $C_{4m-4\Delta}$is completed.

Case 3) m = 2Δ+2

In this case we generate the edge label 4m-4Δ-1 by joining the vertex labeled 4p+Δ-1 to the vertex 4p+4m-3Δ-2. Now we apply transformation type 3 to the vertex labels (4p+3m+2, 4p+3m+1, ... , 4p+3m+1-Δ, 4p+3m-Δ, ... , 4p+2m+2, 4p+2m+1) and (4p+2m, 4p+2m-1, ... , 4p+m+1, 4p+m) by choosing the vertices 4p+3m-Δ+1 and 4p+3m-Δ as end vertices. This transformation yields the remaining edge values and this case is completed.

Case 4) m = 2Δ+1

In this case first we have to modify an initial snake which is constructed at the beginning of the proposition 2. The vertices of this snake in this case successively labeled as follows: (4p+4m+1, 4p, 4p+4m, 4p+1, ... , 4p+4m-Δ+3, 4p+Δ-2, 4p+4m-Δ+2, 4p+Δ, 4p+4m-Δ+1, 4p+Δ+1, 4p+4m-Δ, ... , 4p+m-2, 4p+3m+3, 4p+m, 4p+3m+2, 4p-2). The resulting values of the edges are then 4m+1, 4m, 4m-1, ... , 4m-2Δ+5, 4m-2Δ+4, 4m-2Δ+2, 4m-2Δ+1, 4m-2Δ, ... , 2m+5, 2m+3, 2m +2, 3m+4. The edges labeled 2m+4 and 2m+1 are obtained by joining the following pairs of vertices respectively: 4p+Δ-1 and 4p+2m+Δ+3; 4p+Δ-1 and 4p+2m+Δ. Now we apply transformation type 3 to the rest of vertex labels by selecting the two vertices 4p+2m+Δ+3 and 4p+2m +Δ as end points. Thus the construction of $C_{4m-4\Delta}$ is ended.

Corollary 2: The graph $C_{4a} \cup C_{4b} \cup C_{4c}$ has an α-labeling if (3/2)b < a+ c ≤ 2b-1, a > b > c and a < b +c.

Proof: In proposition 2, we will substitute the parameters as follows: a = p+Δ, b = p and c = m-Δ.

Proposition 3: The graph $C_{4(p+\Delta)} \cup C_{4(m-\Delta)} \cup C_{4p}$ has an α-labeling if 1 < m ≤ (1/2)p and Δ < (2/3)m-1.

***Proof*:** First consider the construction of $2C_{4p} \cup C_{4m}$, 1 < m ≤ (1/2)p, which has been discussed in [3]. Now we remove all the edges of C_{4m} and also the edge (4p, 4p+4m) from the second C_{4p} and try to expand C_{4p} by using the following snake. This snake will have its vertices labeled consecutively as follows: (4p, 4p+4m-1, 4p+3, 4p+ 4m-4, 4p+6, 4p+4m-7, ... , 4p+4m-3Δ+5, 4p+3Δ-3, 4p+4m-3Δ+2, 4p+3Δ, 4p+4m-3Δ, 4p+3Δ-1, 4p+4m-3Δ+3, 4p+3Δ-4, 4p+4m-3Δ+6, ... , 4p+5, 4p+4m-3, 4p+2, 4p+4m). The values of the edges are 4m-1, 4m-4, 4m-7, 4m-10, 4m-13, ... , 4m-6Δ+8, 4m-6Δ+5, 4m-6Δ+2, 4m-6Δ, 4m-6Δ+1, 4m-6Δ+4, 4m-6Δ+7, 4m-6Δ+10, ... , 4m-8, 4m-5, 4m-2. Now we will join the vertex 4p+6m-1 to the vertex 4p+1 and 4p+2m-1 to obtain the edge values 6m-2 and 4m. Next we form another snake in such a way that its vertices will be labeled consecutively by the numbers (4p+1, 4p+4m-2, 4p+4, 4p+4m-5, 4p+7, 4p+4m-8, ... , 4p+3Δ-8,

$4p+4m-3\Delta+7$, $4p+3\Delta-5$, $4p+4m-3\Delta+4$, $4p+3\Delta-2$, $4p+4m-3\Delta+1$). The resulting values of the edges of this snake are then $4m-3$, $4m-6$, $4m-9$, $4m-12$, $4m-15$, ... , $4m-6\Delta+15$, $4m-6\Delta+12$, $4m-6\Delta+9$, $4m-6\Delta+6$, $4m-6\Delta+3$. The edge label $4m-6\Delta-1$ is obtained by joining the two vertices $4p+3\Delta+2$ and $4p+4m-3\Delta+1$. Now we apply transformation type 2 to the rest of vertex labels by choosing the two vertices $4p+3\Delta+2$ and $4p+2m-1$ as end points. The remaining edge values will be generated and the cycle $C_{4(m-\Delta)}$ is completed.

Corollary 3: The graph $C_{4a} \cup C_{4b} \cup C_{4c}$ has an α-labeling if $b+1 < a+c < (3/2)b$, $a > b > c$ and $a < b + c$.

Proof: In proposition 3 assume $a = p+\Delta$, $b = p$ and $c = m-\Delta$. Then we have $p = b$, $\Delta = a-b$ and $m = a-b+c$. This yields that an α-labeling of the graph $C_{4(p+\Delta)} \cup C_{4p} \cup C_{4(m-\Delta)}$ is replaced by $C_{4a} \cup C_{4b} \cup C_{4c}$ with the following conditions:

1) $1 < m \leq (1/2)p \Rightarrow 1 < a-b+c \leq (1/2)b \Rightarrow b+1 < a+c \leq (3/2)b$.

2) $\Delta < (2/3)\ m -1 \Rightarrow (a-b) < (2/3)(a-b+c) -1 \Rightarrow (1/3)(a-b) < (2/3)c -1 \Rightarrow a-b < 2c-3 \Rightarrow a +3 < b+2c$.

We know that $a < b+c$ and if $3 \leq c$ we have $a+3 < b+2c$. Thus for $3 \leq c$ the second condition is an unnecessary condition according to our assumptions. Now we consider the cases $c = 1$ and $c = 2$ as follows:

Case 1: c = 1

$c = 1$, $a < b+c \Rightarrow a < b+1 \Rightarrow a \leq b$ but according to our assumptions $a > b$. Therefore we always have $c > 1$.

Case 2: c = 2

$c = 2$, $a < b+c \Rightarrow a < b+2 \Rightarrow a \leq b+1$ and from $b+1 < a+c \Rightarrow b+1 < a+2 \Rightarrow b < a+1 \Rightarrow b \leq a$. Therefore when $c = 2$ we will have $a = b$ or $a = b+1$ and the second condition is satisfied when $a = b$.
For case $c = 2$, $a = b+1$ we have the graph $C_{4(b+1)} \cup C_{4b} \cup C_8$, $b \geq 1$. We explain the construction of α-labeling of this graph separately:

For $b = 1, 2$ we will have the graph $2C_8 \cup C_4$ and $2C_8 \cup C_{12}$ respectively. The construction of α-labeling of these graphs have been considered in [3]. Now assume that $b > 5$.
The cycle $C_{4(b+1)}$ has the following sequence for the successive vertices: [0, 8b+12, 1, 8b+11, 2, 8b+10, … , 7b+14, b-1, 7b+13, b, 7b+12, b+2, 7b+11, b+3, 7b+10, … , 6b+13, 2b+1, 6b+12, 2b+2, 6b+11]; the resulting values of the edges are then 8b+12, 8b+11, 8b+10, … , 4b+10, 4b+9. The vertices of C_8 will be labeled as follows: [b+1, 5b+9, 3b+6, 5b+6, 3b+9, 5b+5, 3b+7, 5b+8]. The resulting edge values of C_8 are: 4b+8, 4b+7, 2b+3, 2b+2, 2b+1, 2b, 2b-2, 2b-3, 2b-4. The missing value of the whole graph is 2b+3. Now the cycle C_{4b} can be labeled as follows:
1. Form the snake (2b+4, 6b+10, 2b+5, 6b+9, … , 5b+11, 3b+4, 5b+10, 3b+5). The values of the edges are then 4b+6, 4b+5, 4b+4, … , 2b+7, 2b+6, 2b+5.
2. Construct the snake (3b+5, 5b+7, 3b+8, 5b+3). This generates the additional edge labels 2b+2, 2b-1, 2b-5.
3. Join the two vertices labeled 2b+4 and 4b+8 to each other to generate the edge labeled 2b+4.
4. Apply transformation type 3 to the vertex labels (5b+4, 5b+3, … , 4b+8, 4b+7) and (4b+6, 4b+5, … , 3b+11, 3b+10) by considering the two vertices 5b+3 and 4b+8 as end vertices. The corresponding edge values of this transformation will be 2b-6, 2b-7, 2b-8, … , 2, 1 and the cycle C_{4b+1} will be completed.

For b = 3, 4, 5 the construction of an α-labeling of $C_{4(b+1)} \cup C_{4b} \cup C_8$ is given in the table 1 :

b	$C_{4(b+1)} \cup C_{4b} \cup C_8$	Construction of an α-labeling
3	$C_{16} \cup C_{12} \cup C_8$	[0, 35, 1, 34, 2, 33, 3, 32, 5, 31, 6, 30, 7, 29, 8, 36] , [10, 28, 11, 27, 12, 26, 13, 25, 15, 22, 17, 21], [4, 24, 16, 19, 18, 20, 14, 23]
4	$C_{20} \cup C_{16} \cup C_8$	[0, 43, 1, 42, 2, 41, 3, 40, 4, 39, 6, 38, 7, 37, 8, 36, 9, 35, 10, 44] , [12, 34, 13, 33, 14, 32, 15, 31, 16, 30, 17, 27, 20, 23, 22, 24], [5, 28, 19, 25, 21, 26, 18, 29]
5	$C_{24} \cup C_{20} \cup C_8$	[0, 51, 1, 50, 2, 49, 3, 48, 4, 47, 5, 46, 7, 45, 8, 44, 9, 43, 10, 42, 11, 41, 12, 52] , [14, 39, 15, 38, 16, 37, 17, 36, 18, 35, 19, 32, 22, 28, 26, 27, 24, 29, 25, 40], [6, 33, 21, 30, 23, 31, 20, 34]

Table 1: An α-labeling of $C_{4(b+1)} \cup C_{4b} \cup C_8$ for

b =3, 4, 5

3. The Main Theorem

Theorem 1: The graph $C_{4a} \cup C_{4b} \cup C_{4c}$ has an α-labeling for a, b, c $\geq$ 1 with the exception of $3C_4$.

Proof: Table 2 shows a number of special cases of $C_{4a} \cup C_{4b} \cup C_{4c}$. The last column of this table is refer to the references that you can find the construction of an α-labeling of the relative graph:

Case	$C_{4a} \cup C_{4b} \cup C_{4c}$	Reference
a = b = c	$3C_{4a}$	see [3]
a = b	$2C_{4a} \cup C_{4c}$	see [3]
a $\geq$ b + c	$C_{4a} \cup C_{4b} \cup C_{4c}$	see [1]

Table 2: References for construction of an α-labeling of special classes of the graph $C_{4a} \cup C_{4b} \cup C_{4c}$

Now without any loss of generality assume $a < b + c$, $b < a + c$, $c < a + b$ and $a > b > c$. Now we have the following result:
$c < b \Rightarrow 2c < 2b$ & $a < b + c \Rightarrow a+2c < 3b + c \Rightarrow a + c < 3b$
In the other hand if $a + c = b + 1 \Rightarrow c = b - a + 1$ but we have $a > b$ thus $c \leq 0$ which is impossible so $b+1 < a + c$. In other words based on these assumptions we will have $b+1 < a + c < 3b$. The construction of an α-labeling of the graph $C_{4a} \cup C_{4b} \cup C_{4c}$ when $b+1 < a + c < 3b$, $a < b + c$ and $a > b > c$ has been developed in the former corollaries as shown in the following table 3:

Case	The construction of an α-labeling
$b+1 < a + c \leq (3/2)b$,	see corollary 3
$(3/2) b < a + c \leq 2b-1$	see corollary 2
$2b \leq a + c < 3b$	see corollary 1

Table 3: References for construction of α- labelings of special cases of the graph $C_{4a} \cup C_{4b} \cup C_{4c}$ when $a < b + c$ and $a > b > c$

References:

1. J. Abrham and A. Kotzig, Graceful valuations of 2-regular graphs with two components, *Discrete Math.* , Vol 150, 1996, pp. 3-15.
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3. K. Eshghi and M. Carter, *Topics in Applied and Theoretical Mathematics*, Wseas Press, 2001, pp. 139-155.
4. A. Rosa, On certain valuations of the vertices of a graph, *Theory of Graphs*, Gordon and Breach, 1967, pp. 349-355.