Proper and Non-singular Algebraic Numerical Algorithms

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Abstract: - Proper and coherent conditions for Algebraic Numerical Algorithms (ANA) and Algebraic Finite Difference Equation (AFDCE) are treated as a realistic tool. Importance of algebraic finiteness that is generated by blowing-up is shown relating ANA's and AFDCE's integrability.

Key-Words: - Algebraic finite difference equation, proper condition, blowing-up

1 Introduction

Proper and coherent conditions for ANA and AFDCE in [1, 2] are revised as realistic tool. In previous work, newly introduced conditions are fairly abstractive to actual treatment. Therefore actual treatments as for these conditions are shown in this article. First we shortly review coherent and proper conditions. Next realistic treatment is explained using 2-step AFDCE as a sample.

2 Coherent sheaves by AFDCE in affine space

Consider following simple 2-step algebraic finite difference equation as an example,

(1) $$F(f_{n-1}, f_n, f_{n+1}) = 0,$$

here n is integer and $F(f_{n-1}, f_n, f_{n+1}) \in C[f_{n-1}, f_n, f_{n+1}]$. $C[f_{n-1}, f_n, f_{n+1}]$ is polynomial function by $\{f_{n-1}, f_n, f_{n+1}\}$ with complex coefficient C. We write C_n=$C[f_{n-1}, f_n, f_{n+1}]$ and $F_n = F(f_{n-1}, f_n, f_{n+1})$. Here f_j=$f(z_j)$, $z_j \in C$, and j is order of the sequence of points $\{\ldots, z_{j-1}, z_j, z_{j+1}, z_{j+2}, \ldots\}$. Then we can regard (1) as functional equation of $f(z)$.

Discrete analogy for sheaves of modules in AFDCE to usual affine scheme can be obtained as,

(i) Assume every F_j corresponds to prime ideals.

(ii) We put S_n=$C_n \backslash F_n$ and A_n= $S_n^{-1}C_n$. Then A_n is Noetherian at least locally.

(iii) Treat each $\phi_n^F : A_n \to A_{n+1}$ as homomorphism by natural morphism $\{f_{n-1}, f_n, f_{n+1}\} \to \{f_n, f_{n+1}, f_{n+2}\}$. In the same manner we treat $\phi_n^B : A_{n+1} \to A_n$. Here ϕ_j^F and ϕ_j^B means forward and backward evolutional scheme at step j.

(iv) We define X_n=$SpecA_n$ and X_n as all prime ideal of A_n. Define (X, A)={Collection of all (X_j, A_j)}.We introduce Zariski topology by open covering U_j and D_j that are defined as U_j=$\{p \mid f_j \notin p, p \in X\}$ and D_j=$\{p \mid F_j \notin p, p \in X\}$. We find X_j is Noetherian locally because A_j is Noetherian.

Using above definitions, we can introduce sheaves of AFDCEs. It is known that sheaves by ideals become coherent sheaves. Proper scheme over C, which is coherent sheaf, corresponds to some analytical scheme by GAGA [1, 2]. Especially projective scheme over C is proper scheme.

Since $n \in Z$, collection of all A_n and C_n=$C[f_{n-1}, f_n, f_{n+1}]$, $C[f_n, f_{n+1}, f_{n+2}]$, ..., $C[f_{k-1}, f_k, f_{k+1}]$ are polynomials and consist of infinite number of variables. That is, $SpecA$ has infinite elements and is not Noetherian. Above implementation (i) to (iv) satisfies coherent sheaf condition only locally (at every n). Therefore we need more conditions to construct entire coherent sheaf of AFDCE by this formulation for the integrable numerical scheme.

For the condition of finite number of variables in entire space, we must add more conditions to AFDCE. For example, forcing following condition gives Noetherian property of entire space of AFDCE (1),

(2) $$\begin{vmatrix} \dfrac{\partial F_n}{\partial f_{n-1}} & \dfrac{\partial F_n}{\partial f_{n+2}} \\ \dfrac{\partial F_{n+1}}{\partial f_{n-1}} & \dfrac{\partial F_{n+1}}{\partial f_{n+2}} \end{vmatrix} = \left| \frac{\partial F_n}{\partial f_{n-1}} \frac{\partial F_{n+1}}{\partial f_{n+2}} \right| \neq 0, \text{ for all } n,$$

or equivalently

(3) $$\left| \frac{\partial F_n}{\partial f_{n-1}} \frac{\partial F_n}{\partial f_{n+1}} \right| \neq 0, \text{ for all } n.$$

By implicit function theorem and (3), we can find following local relations at every n,

(4) $f_{n+1}=g_{n(+)}(f_n, f_{n-1}), f_{n-1}=g_{n(-)}(f_n, f_{n+1}),$

here $g_{n(+)}$, $g_{-n(-)}$ should be function that never spoil algebraic property of each F_n. Then we can delete f_{n-1} and f_{n+2} from $C[f_{n-1}, f_n, f_{n+1}]$, $C[f_n, f_{n+1}, f_{n+2}]$ as $C[g_{n(-)}(f_n, f_{n+1}), f_n, f_{n+1}]$, $C[f_n, f_{n+1}, g_{n+1(+)}(f_n, f_{n+1})]$. Appling this condition to all F_n, we find all $C[f_{n-1}, f_n, f_{n+1}]$, $C[f_n, f_{n+1}, f_{n+2}]$,..., $C[f_{k-1}, f_k, f_{k+1}]$,... are included in the two variable polynomial $C[f_k, f_{k+1}]$ or holomorphic function. Since k is arbitrary, we can say $C[f_k, f_{k+1}]$ is germ at $(k, k+1)$ and also representation of solution function of AFDCE by germ at $(k, k+1)$. Expression of $C[f_j, f_{j+1}]$ by $C[f_k, f_{k+1}]$, $j \neq k$ is analogous to Taylor series representation $C[f_j, f_{j+1}]$ by $\{f_k, f_{k+1}\}$. In this case it is functional series representation for near neighbor functions.

Definition 1: For general ADFCE, we define coherent condition as,

(i) F_n gives coordinate ring, and F_n generates coverings of AFDCE as a non-singular algebraic manifold. Moreover A_n becomes Noetherian at every n.

(ii) Existence of proper morphism ϕ_n^F for forward evolution and ϕ_n^B for backward evolution, and if necessary both of them, at every n. In addition every A_n satisfies coherent condition by Zariski topology.

(iii) Following dimensional condition is satisfied independently of n in each covering with regular coordinate system. dim(A_n)=dim(Initial conditions or Boundary conditions) = Const.

Definition 2: We call singular point (set) of AFDCE where coherent condition is broken.

It is clear from the definition that CAFDCE has no singular points (set), because singular set by normal algebraic definition is included in singular set of AFDCE. In other words, non-singular AFDCE is non-singular algebraic manifold with proper local coordinates. Therefore if AFDCE is defined as singular algebraic equation, we need to modify it to non-singular one.

3 Blowing-up and parameterization for single-step AFDCE

Singularity of the AFDCE is removable by blowing-up or some resolution procedure. As the simple example we consider the curve

(5) $F(x, y)=y^2 - x^3 - Cx^2=0,$

here C is parameter or constant. The curves when $C=-1, 0, 1$ are shown in Fig. 1 (a), (b), (c).

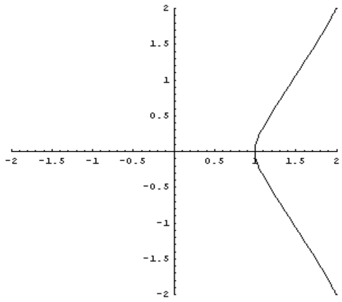

Fig. 1 (a) $y^2 - x^3 - Cx^2=0, C=-1$

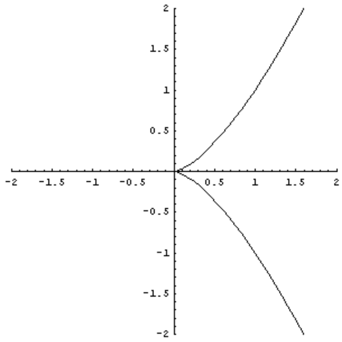

Fig. 1 (b) $y^2 - x^3 - Cx^2=0, C=0$

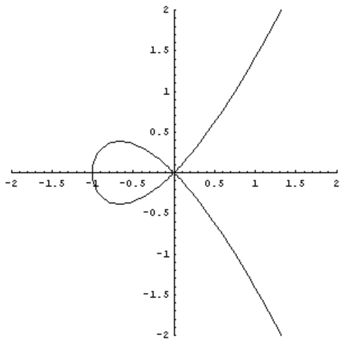

Fig. 1 (c) $y^2 - x^3 - Cx^2=0, C=1$

Singular point of the curve is given by

(6) $$\frac{\partial F}{\partial y} = \frac{\partial F}{\partial x} = 0 \to y = 0, x(3x - 2C) = 0 \cdot$$

Easily we know $(x, y)=(2C/3, 0)$ is not on the curve, except for $C=0$. Therefore $(x, y)=(0, 0)$ is a singular point. We discretize (5) by elimination of C. Then we get,

(7) $$\frac{y_{n+1}^2 - x_{n+1}^3}{x_{n+1}^2} = \frac{y_n^2 - x_n^3}{x_n^2} = C$$

By deformation for y_{n+1} we get AFDCE,

(8) $$y_{n+1}^2 = (y_n^2 - x_n^3)\frac{x_{n+1}^2}{x_n^2} + x_{n+1}^3$$

We know (8) is multi-value function and initial condition defines singular property of the solution curve. We also find $x<-C$ is meaningless for evaluation of y. This condition is usually difficult to find out from (8) at the first glance. It is clear that parameterization for each x and y is necessary in this case. Fortunately it is easy to find parameterization for the curve, because curve is one dimension. Blowing-up gives well posed local parameter. In general higher dimensional case, blowing-up gives negative behavior. It will be shown in later example.

Before introducing blowing-up, we discuss the singularity of (8) from AFDCE point. Simple question is how we can find the same singularity to (5) from (8) directly. It should be made clear first. We define

(9)

$$F_{n+1}(x_n, x_{n+1}, y_n, y_{n+1}) = y_{n+1}^2 - (y_n^2 - x_n^3)\frac{x_{n+1}^2}{x_n^2} - x_{n+1}^3 .$$

By implicit function theorem, (8) should satisfy

(10) $$\frac{\partial F_{n+1}}{\partial y_{n+1}} \neq 0 \text{ or } \frac{\partial F_{n+1}}{\partial x_{n+1}} \neq 0$$

for the mapping from n to $n+1$(forward evolutional case). Contrary, we can define singularity by

(11) $$\frac{\partial F_{n+1}}{\partial y_{n+1}} = 0 \text{ and } \frac{\partial F_{n+1}}{\partial x_{n+1}} = 0 .$$

Then $\partial F_{n+1} / y_{n+1} = 0$ gives $y_{n+1} = 0$, and $\partial F_{n+1} / x_{n+1} = -x_{n+1}(2(y_n^2 - x_n^3)/x_n^2 - 3x_{n+1}) = 0$ gives $x_{n+1}=0$ or

(12) $$x_{n+1} = 2/3 \cdot (y_n^2 - x_n^3)/x_n^2 = 2/3 \cdot C .$$

When $y_{n+1}=0$, (8) gives $x_{n+1}=0$ or $x_{n+1}=-C$, therefore singular point is only $(x_{n+1}, y_{n+1})=(0, 0)$. Singular point is isolate. It found that initial conditions (x_n, y_n) which satisfy

(13) $$C = \frac{y_n^2 - x_n^3}{x_n^2} \geq 0 ,$$

give singularity to the curve at the origin. This is simple example which illustrates dependency of the singularity on the initial conditions of AFDCE.

Now we use blowing-up [3, 4]. In this case (x_{n+1}, y_{n+1}) is the center of blowing-up and becomes exceptional set. We treat origin with projective space $\{\xi;\eta\} \in P^1, (\xi,\eta) \neq (0,0)$ as $(\mathrm{x}_{n+1}, \mathrm{y}_{n+1}) \times \{\xi_{n+1};\eta_{n+1}\}$, and $x_{n+1}\eta_{n+1} = y_{n+1}\xi_{n+1}$. Then we may use $x_n\eta_n = y_n\xi_n$ also implicitly. Note that it is no difference if we use (x_n, y_n) as the center of blowing-up instead of (x_{n+1}, y_{n+1}), because (7) is symmetric as for (x_n, y_n) and (x_{n+1}, y_{n+1}). The condition (10) is for the forward difference scheme. For the backward difference scheme we must use

(14) $$\frac{\partial F_{n+1}}{\partial y_n} = 0, \frac{\partial F_{n+1}}{\partial x_n} = 0 .$$

This backward evolutional condition gives that (x_n, y_n) is a singular point. Moreover if scheme is reversible, we must treat both cases.

We define two cover U_1 ($\eta_{n+1} \neq 0$) and U_2($\xi_{n+1} \neq 0$). We also assume $\eta_n \neq 0$ in U_1, and $\xi_n \neq 0$ in U_2 by continuity of variables η and ξ . In U_1, substitution $x_{n+1} = y_{n+1}\xi_{n+1}/\eta_{n+1}, x_n = y_n\xi_n/\eta_n$ into (8) gives,

(15) $$y_{n+1}y_n = 0 \text{ , or}$$

(16) $$y_{n+1}\xi_{n+1} - y_n\xi_n = \frac{\xi_n^2 - \xi_{n+1}^2}{\xi_n^2\xi_{n+1}^2}, x_{n+1} = y_{n+1}\xi_{n+1} .$$

Here we put $\eta_n = \eta_{n+1} = 1$.

By $x_{n+1} = y_{n+1}\xi_{n+1}/\eta_{n+1}, x_n = y_n\xi_n/\eta_n$ and $\xi_n = \xi_{n+1} = 1$ in U_2, (8) becomes

(17) $$x_{n+1}x_n = 0 \text{, or}$$

(18) $$x_{n+1} - x_n = \eta_{n+1}^2 - \eta_n^2, y_{n+1} = x_{n+1}\eta_{n+1} .$$

It is clear (15) and (17) equal $(x_{n+1}, y_{n+1})=(0, 0)$, because of $y_n \neq 0$ and $x_n \neq 0$. Then (15) and (17) correspond to original center of blowing-up. Equations (16) and (18) give other representations of (8) in each U_1 and U_2 at the near origin using parameter ξ and η . Clearly (18) gives better representation of (8). Of course exchanging order of above procedures, blowing-up (8) first and discretizing it next give the same equations.

Last sample in this section is AFDCE which is obtained by eliminating integral constants in general solution. Consider following general solution y_g ,

(19) $$F(y_s, y_g) = y_g^2 - C_1 y_s^3 - C_2 y_s^2 = 0$$

Here C_1 and C_2 are integral constants, and y_s consist form combination of homogeneous and particular solution functions. If we teat $C_1=1$, $C_2=C$, $y_g=y$ and $y_s=x$ then (19) is the same to (5). In this case the general solution is parameterized properly by y_s with parameters introduced by previous blowing-up procedure. We get corresponding AFDCE by elimination of C_1 and C_2 from (19). Then we can use blowing-up to the AFDCE for desingularization and introduce proper parameters for integrable discretization. By further consideration we can respect that using blowing-up like this manner for desingularization gives proper parametric coordinate for the AFDCE of that form $K[y, y_1, y_2, \ldots, y_n, C_1, C_2, \ldots, C_n] = 0$. Here K[*] means K field coefficient * variable polynomials. It is known that blowing-up gives proper local parameter if it is one dimensional case (curve) generally. Blowing-up for higher dimensional singularity gives simpler singular manifold again in general. Then after finite number of blowing-up, we can eliminate singularities from it.

4 Projective scheme in AFDCE

We introduced coherent condition into AFDCE in the previous section. One more condition is necessary to use GAGA for the integrability of AFDCE. A condition is proper morphism property of AFDCE. It is known that morphism in projective space is proper morphism, therefore we don't need to pay attention to this property when we treat AFDCE in projective space. In this section we review projective scheme shortly for this purpose.

We assume all AFDCEs in this section are homogeneous equations. As an example, using the same notation in previous section, we treat F_n in C_n. In this case $C_n = C[f_{n-1}; f_n; f_{n+1}; f_{o,n}]$ corresponds to polynomial function with complex coefficient in projective space. Then F_n is defined as,,

(20) $\{f_{n-1}, f_n, f_{n+1}\} \xrightarrow[\text{bijection}]{} \{\frac{f_{n-1}}{f_{0,n}}; \frac{f_n}{f_{0,n}}; \frac{f_{n+1}}{f_{0,n}}; 1\}$,

here $f_{0,n} \neq 0$, $\{0;0;0;0\} \notin \{f_{n-1}; f_n; f_{n+1}; f_{0,n}\}$ and

(21) $F_n \to (f_{0,n})^m F(\frac{f_{n-1}}{f_{0,n}}, \frac{f_n}{f_{0,n}}, \frac{f_{n+1}}{f_{0,n}})$, $0 \leq m$,

when total order of $F(f_{n-1}, f_n, f_{n+1})$ equals m. By this treatment we can regard F_n in projective space as,

(22) $B_n = (f_{0,n})^m F(\frac{f_{n-1}}{f_{0,n}}, \frac{f_n}{f_{0,n}}, \frac{f_{n+1}}{f_{0,n}})$

$= G_n(f_{n-1}; f_n; f_{n+1}; f_{0,n})$,

here $B_n \in C[f_{n-1}; f_n; f_{n+1}; f_{o,n}]$ is homogeneous equation, and $F_n = G_n(f_{n-1}; f_n; f_{n+1}; 1)$. For simplicity we assume B_n is a homogeneous prime ideal. We consider a space $Proj(PA_n)$ which consists of all homogeneous prime ideals except for irrelevant ideal in quotient ring $PA_n = S_n^{-1} C_n$, here $S_n = C[f_{n-1}; f_n; f_{n+1}; f_{o,n}] \setminus B_n$. We call this space $PX_n = Proj(PA_n)$. In the same manner in affine space, we can introduce Zariski topology locally using following definitions for open covering,

(23) $D_j = \{p \mid PA_j \notin p, p \in PX\}$,

here $PX = \{$Collection of all $Proj(PA_j)\}$. We also use affine covering U_j to cover D_i. In this case set of U_j is finer covering than set of D_i.

We can treat inclusion F_n to projective space by different way from previous example, as following.

(24) $\{f_{n-1}, f_n, f_{n+1}\} \xrightarrow[\text{bijection}]{} \{\frac{f_{n-1}}{f_{0,n-1}}; \frac{f_n}{f_{0,n}}; \frac{f_{n+1}}{f_{0,n+1}}; 1; 1; 1\}$,

here $f_{0,n-1} \neq 0$, $f_{0.n} \neq 0$, $f_{0.n+1} \neq 0$,

$\{0;0;0;0;0;0\} \notin \{f_{n-1}; f_n; f_{n+1}; f_{0,n-1}; f_{0,n}; f_{0,n+1}\}$

and

(25) $F_n \to$

$(f_{0,n-1})^{m1}(f_{0,n})^{m2}(f_{0,n+1})^{m3} F(\frac{f_{n-1}}{f_{0,n-1}}, \frac{f_n}{f_{0,n}}, \frac{f_{n+1}}{f_{0,n+1}})$

$= G_n(f_{n-1}; f_n; f_{n+1}; f_{0,n-1}; f_{0,n}; f_{0,n+1})$,

$0 \leq m1, m2, m3$, order of f_{n-1}, f_n, f_{n+1} in F.

Since $Proj(PA_n)$ is coherent sheaf at every n locally, we must add more condition to PX which becomes coherent sheaf globally in addition to (2). At the first we must define a rule how to choose proper $f_{0,j}$ for all j. Clearly we have no rule yet for selecting $f_{0,j}$ for all j. We must hold the total number of $f_{0,j}$ in finite, because its number relates the number of initial conditions. It maybe better choice for $f_{0,j}$ to make PA_n non-singular algebraic manifold. Therefore $f_{0,j}$ is defined by blowing-up at each j. We must never forget the total number of $f_{0,j}$ is finite even if we define it by blowing-up. Note that proper condition and non-singular condition are different.

5 Modification of AFDCE as proper system, a realized sample

Following idea in previous section, we consider the actual treatment of artificially introduced variables $\{f_{n-1}; f_n; f_{n+1}; f_{0,n-1}; f_{0,n}; f_{0,n+1}\}$ in (24) and (25). Using these variables, (1) become proper scheme. For the realization of the idea, we treat (1) in 3-dimensional real algebraic torus, defined as

$R[f_{n-1}, f_n, f_{n+1}, g_{n-1}, g_n, g_{n+1}]$, $f_{n-1}=1/g_{n-1}$, $f_n=1/g_n$, $f_{n+1}=1/g_{n+1}$. Roughly, $\{f_j; f_{0,j}\}$ in (24) correspond to $\{f_j; g_j\}$ in this treatment.

In this space we can treat (1) as following equations which are all equivalent.

In $U_1=(g_{n-1}\neq 0, g_n\neq 0, g_{n+1}\neq 0) = (f_{n-1}\neq\infty, f_n\neq\infty, f_{n+1}\neq\infty)$,

(26) $F(f_{n-1}, f_n, f_{n+1})=0$.

In $U_2=(f_{n-1}\neq 0, g_n\neq 0, g_{n+1}\neq 0) = (g_{n-1}\neq\infty, f_n\neq\infty, f_{n+1}\neq\infty)$,

(27) $$F(1/g_{n-1}, f_n, f_{n+1})=0 \\ \to g_{n-1}^p F(1/g_{n-1}, f_n, f_{n+1})=0,$$

here p is the order of f_{n-1} in (1).

In $U_3=(g_{n-1}\neq 0, f_n\neq 0, g_{n+1}\neq 0) = (f_{n-1}\neq\infty, g_n\neq\infty, f_{n+1}\neq\infty)$,

(28) $$F(f_{n-1}, 1/g_n, f_{n+1})=0 \\ \to g_n^q F(f_{n-1}, 1/g_n, f_{n+1})=0,$$

q is the order of f_n in (1).

In the same manner:

$U_4=(g_{n-1}\neq 0, g_n\neq 0, f_{n+1}\neq 0) = (f_{n-1}\neq\infty, f_n\neq\infty, g_{n+1}\neq\infty)$,

(29) $$F(f_{n-1}, f_n, 1/g_{n+1})=0 \\ \to g_{n+1}^r F(f_{n-1}, f_n, 1/g_{n+1})=0;$$

r is the order of f_{n+1} in (1).

$U_5=(f_{n-1}\neq 0, f_n\neq 0, g_{n+1}\neq 0) = (g_{n-1}\neq\infty, g_n\neq\infty, f_{n+1}\neq\infty)$;

(30) $$F(1/g_{n-1}, 1/g_n, f_{n+1})=0 \\ \to g_{n-1}^p g_n^q F(1/g_{n-1}, 1/g_n, f_{n+1})=0;$$

$U_6=(f_{n-1}\neq 0, g_n\neq 0, f_{n+1}\neq 0) = (g_{n-1}\neq\infty, f_n\neq\infty, g_{n+1}\neq\infty)$,

(31) $$F(1/g_{n-1}, f_n, 1/g_{n+1})=0 \\ \to g_{n-1}^p g_{n+1}^r F(1/g_{n-1}, f_n, 1/g_{n+1})=0;$$

$U_7=(g_{n-1}\neq 0, f_n\neq 0, f_{n+1}\neq 0) = (g_{n-1}\neq\infty, f_n\neq\infty, f_{n+1}\neq\infty)$,

(32) $$F(f_{n-1}, 1/g_n, 1/g_{n+1})=0 \\ \to g_n^q g_{n+1}^r F(f_{n-1}, 1/g_n, 1/g_{n+1})=0;$$

In $U_8=(f_{n-1}\neq 0, f_n\neq 0, f_{n+1}\neq 0) = (g_{n-1}\neq\infty, g_n\neq\infty, g_{n+1}\neq\infty)$,

(33) $$F(1/g_{n-1}, 1/g_n, 1/g_{n+1})=0 \\ \to g_{n-1}^p g_n^q g_{n+1}^r F(1/g_{n-1}, 1/g_n, 1/g_{n+1})=0.$$

Using above coverings $U_j, j=1,\ldots,8,$ we can treat original AFDCE (1) as proper AFDCE.

In this article we call pole singularity at j when $f_j=\infty$ in (1). We must treat triple (29), (28) and (27) when (1) crosses isolate single pole singularity. Sequence (29), (32), (30), (27) also occur when orbit crosses two isolate pole singularities. We can list up easily possible combinations of sequence of coverings that are proper for treating sequence of pole singularities. Of course we can also imagine the case that these sequences occur periodically. If we investigate whole possible singularity with arbitrary initial conditions, we must investigate all coverings. There is sometimes no necessity to investigate all singularities in all coverings when we integrate AFDCE with specific initial conditions. Here we consider general case.

We treat (1) in each U_j as,

(26)' $G(f_{n-1}, f_n, f_{n+1}) = F(f_{n-1}, f_n, f_{n+1})=0$,

(27)' $G(g_{n-1}, f_n, f_{n+1}) = g_{n-1}^p F(1/g_{n-1}, f_n, f_{n+1})=0$,

(28)' $G(f_{n-1}, g_n, f_{n+1}) = g_n^q F(f_{n-1}, 1/g_n, f_{n+1})=0$,

(29)' $G(f_{n-1}, f_n, g_{n+1}) = g_{n+1}^r F(f_{n-1}, f_n, 1/g_{n+1})=0$,

(30)' $G(g_{n-1}, g_n, f_{n+1}) = g_{n-1}^p g_n^q F(1/g_{n-1}, 1/g_n, f_{n+1})=0$,

(31)' $G(g_{n-1}, f_n, g_{n+1}) = g_{n-1}^p g_{n+1}^r F(1/g_{n-1}, f_n, 1/g_{n+1})=0$,

(32)' $G(f_{n-1}, g_n, g_{n+1}) = g_n^q g_{n+1}^r F(f_{n-1}, 1/g_n, 1/g_{n+1})=0$,

(33)' $G(g_{n-1}, g_n, g_{n+1}) = g_{n-1}^p g_n^q g_{n+1}^r F(1/g_{n-1}, 1/g_n, 1/g_{n+1})=0$.

Hereafter we use notation

(34) $G_{j,n} \equiv G(y_{n-1}, y_n, y_{n+1})=0, j=1,\ldots,8.$

for abbreviation which is the same as some equations in $U_j, j=1,\ldots,8$ at n. For example, $\{y_{n-1}, y_n, y_{n+1}\}$ in $G_{2,n}$ equal to $\{g_{n-1}, f_n, f_{n+1}\}$. It is clear that all $G_{j,n}$ should be non-singular. Because we assume initial conditions are arbitrary, we cannot specify which cover includes orbit. In this system we can treat pole singularities and algebraic singularities at the infinity in (1) as singularities at the near origin using appropriate coverings.

6 Desingularization of the proper AFDCE, a sample

We remove singularity of the ADFCE by blowing-up or resolution procedure. As an example we consider proper system (34) obtained from original AFDCE. As for singularities which must be removed, we can consider following situations:

1. After finite number of advancing step using the given initial conditions, its orbit falls on or crosses singular points or set;
2. We can't predict the exact location and structure of singularity from arbitrariness of the initial

conditions. In this case we can say AFDCE has moving singularity.

Discrete evolutional equation has remarkable property that the equation suffers no effect of singularity except that the orbit fall on or cross the singular points or set. In other words, we need at least to remove singularity from the orbit. Therefore we must regard two cases as the same problem. It is sufficient that we only consider singularity near the origin in each covering U_j.

We must pay more attention for the following complexities by the blowing-up;

1. What kinds of singularities are included in (34)? These singularities are isolate, periodic or convergent, compact or not.
2. How orbit crosses the singularities?
3. Whether divergence of number of variables, varieties of algebra or coverings by blowing up for singularities of (34) occur or not when advancing integration steps.

For these complexities, finiteness of algebraic relation for (34) from coherent condition claims following strict conditions. The centers of the blowing-up are all compact subset of (34) and algebraically finite. Here algebraically finite means number of the algebraic relations is finite to define the variety (manifold). In this case centers of the blowing-up correspond to variety (manifold). Referring the removal of singularities defined by finite number of algebraic relations is out of scope of this short article, therefore we show simple example. Consider

(35)

$$G_{k,n}(y_{n-1}, y_n, y_{n+1}) \equiv$$
$$y_{n+1}y_{n-1}f_1(y_n) + f_2(y_n)(y_{n+1} + y_{n-1}) + f_3(y_n) = 0$$

n: integer, f_j, $j=1,2,3$ are analytic function of y_n. We treat (35) in

$X \times P^2(C) = (y_{n-1}, y_n, y_{n+1}) \times \{u; v; w\}$ with

$y_{n-1}v - y_n u = 0, y_{n-1}w - y_{n+1}u = 0,$

$y_n w - y_{n+1}v = 0$, here u, v, w and X are defined as

(36) $u = \dfrac{\partial G_{k,n}}{\partial y_{n-1}}, v = \dfrac{\partial G_{k,n}}{\partial y_n}, w = \dfrac{\partial G_{k,n}}{\partial y_{n+1}}$,

$X = \{(y_{n-1}, y_n, y_{n+1}) \in A^3 \mid u = v = w = 0\}$. Clearly $u=v=w=0$ gives algebraic singular points or set. On the other hand singular points or set of (35) are given by $w=0$ for forward AFDCE , $u=0$ for backward AFDCE and $u=w=0$ for reversible AFDCE. These conditions are given by connections between coverings respect to advancing integration step n to $n+1$ or n to $n-1$. We also define coverings,

(37) $U_1 = (y_{n-1}, u_1 = v/u, u_2 = w/u), u \neq 0$.

(38) $U_2 = (y_n, v_1 = u/v, v_2 = w/v),\ v \neq 0$.

(39) $U_3 = (y_{n+1}, w_1 = u/w, w_2 = v/w),\ w \neq 0$.

By these preparations we found that: Backward AFDCE is not singular in U_1, because of $u \neq 0$; Forward AFDCE is not singular in U_3, because of $w \neq 0$; Reversible AFDCE in not singular in $U_1 \cap U_3$. As a result, for example, singularities for forward AFDCE are covered by U_1 and U_2. If (35) in U_1 and U_2 have singularities again for u_2 or v_2, we must repeat blow-up.

Hereafter we consider forward AFDCE only. Then $w=0$ of (35) is,

(36) $y_{n-1}f_1(y_n) + f_2(y_n) = 0$.

If (36) is satisfied by infinite number of points, we can not construct finite number of coverings by blowing-up. As a result this situation breaks coherent condition.

This simple example makes clear the following conclusion. Total number of the algebraic relation of AFDCE which satisfies coherent condition should be finite count in relations generated by blowing-up through the numerical integration. In other words if AFDCE does not satisfy this condition, the AFDCE is not integrable from GAGA point. Please remember that number of algebraic relation generated by blowing-up depend on the number of singularities and the character of singularities also. This result partially supports conjecture in [5].

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