Exciting Forces due to Diffraction of Water Waves on a Sphere in Finite Depth Water

SWAROOP NANDAN BORA
Department of Mathematics
Indian Institute of Technology Guwahati
North Guwahati, Guwahati 781039, Assam
INDIA
swaroop@iitg.ernet.in   http://www.iitg.ernet.in/scifac/swaroop

Abstract: - Laplace's equation is of immense application in fluid dynamics in general and in water wave problems in particular. The evaluation of hydrodynamic coefficients and loads on submerged bodies has a lot of significance in designing these structures. Analytical expressions for the incident potential, diffraction potential and exciting forces due to the effects of diffraction arising out of interaction of water waves with a submerged sphere are derived. Theory of multipole expansions is used in obtaining the velocity potential in terms of an infinite series of associated Legendre polynomials with unknown coefficients. Two motions, namely surge and heave motions, are considered. Numerical results for the exciting forces are presented in tabular form for various depth to radius ratios.

Key-Words: - Laplace's equation, Diffraction, Multipole expansion, Surge, Heave, Associated Legendre polynomial, Exciting forces.

1 Introduction

The forces exerted by the surface waves on a structure in water are very important for designing these structures. Accurate prediction of wave loads becomes indispensable in order to design safe structures. The researchers have been trying to evaluate the various loads and coefficients associated with the interaction of water waves with a submerged sphere.

A number of notable works have been done over the last few decades on analytical solutions for the linear or first order forces acting on a floating or submerged body of spherical, hemispherical or spheroidal shape in water. Havelock can be considered the pioneer in the area of hydrodynamic loading on spherical structures. Havelock [1] started with calculating the wave resistance of a submerged spheroid by replacing it with a distribution of sources and sinks, or of doublets.

Hulme [2] considered heave and surge motions of a floating hemisphere to derive added-mass and damping coefficients associated with the periodic motions. Wang [3] discussed the free motions of a submerged vehicle with spherical hull form but with different metacentric heights. The works of Hulme and Wang were based on the multipole expansions of Thorne [4] which proves to be very successful for periodic motions without forward speed but this method does not seem to be applicable to the problem of a body with forward speed. Wu and Eatock Taylor [5] considered a submerged sphere advancing in regular deep water waves at a constant forward speed. The solution cleared the doubts about the influence of forward speed on hydrodynamic forces. Wu and Eatock Taylor [6] considered a submerged sphere moving in a circular path at constant angular velocity, the analysis being based on the linearized velocity potential theory. Wu *et al.* [7] presented a solution for the wave induced drift forces acting on a submerged sphere in a finite water depth based on a linearized potential theory. Bora [8] considered a sphere in finite depth water and used multipole expansions method to solve the diffraction and radiation problem for surge, heave and pitch motions.

In this paper we present an analytical procedure for the boundary value problem to evaluate the exciting forces due to diffraction for a submerged sphere in finite depth water for surge and heave motions. We consider the boundary value problem to consist of Laplace's equation with a number of appropriate boundary conditions. The diffraction velocity potential is expressed in terms of an infinite series of associated Legendre polynomials with unknown coefficients. Using the body boundary condition, we set up a linear system of equations. By solving the linear system, we can find the velocity potential, and hence the exciting forces along horizontal and vertical directions can be evaluated.

2 Problem Formulation

We assume that the fluid is homogeneous, inviscid and incompressible, and the fluid motion is irrotational. The waves are also assumed to be of small amplitude. Here we consider the diffraction-related hydrodynamic loading with two degrees of freedom, namely, the two translational motions in the x and z directions, i.e. surge and heave motions respectively. We consider a surface wave of amplitude A and wave frequency σ incident on a sphere of radius a submerged in water of finite depth d. The wave is parallel to the x-axis at the time of incidence on the sphere and is propagating along the positive direction.

We consider two sets of coordinate systems. One is a right-handed Cartesian coordinate system (x, y, z), in which the xy plane coincides with the undisturbed free surface and the z-axis is taken vertically downwards from the still water level. The other coordinate system is the spherical coordinate system (r, θ, ψ) with the origin at the geometric centre of the sphere. The centre of the sphere is taken at $(0, 0, h)$ with respect to the Cartesian coordinate system.

The relationship between the coordinate systems is :

$$\begin{aligned} R &= \sqrt{x^2+y^2},\ r=\sqrt{R^2+(z-h)^2} \\ \tan\theta &= \frac{R}{z-h} \qquad \text{for } 0 \le \theta \le \pi \\ \tan\psi &= \frac{y}{x} \qquad \text{for } -\pi \le \psi \le \pi \end{aligned}$$

For an incompressible and inviscid fluid, and for small amplitude wave theory with irrotational motion, we can express the fluid motion by introducing a velocity potential $\Phi(r, \theta, \psi, t)$. This Φ can be written as:

$$\Phi(r,\theta,\psi,t) = Re[\phi(r,\theta,\psi)e^{-i\sigma t}] \qquad (1)$$

where Re stands for the real part.

The motion is assumed harmonic. Also, from Bernoulli's equation, we get pressure, $P(r, \theta, \psi, t)$, as

$$P = -\rho\frac{\partial\Phi}{\partial t} \qquad (2)$$

where ρ is the density of water.

In formulating the diffraction problem, the linearity of the situation enables us to decompose the time-independent velocity potential $\phi(r, \theta, \psi)$ into two velocity potentials ϕ_I and ϕ_D where ϕ_I is the incident potential and ϕ_D is the velocity potential due to the diffraction of the incident wave acting on the sphere. Thus ϕ can be written as $\phi = \phi_I + \phi_D$.

To obtain the velocity potential ϕ, the following boundary problem is to be solved:

1) Laplace's equation in spherical coordinates:

$$\nabla^2\phi = 0 \qquad (3)$$

2) Free surface condition:

$$\frac{\partial\phi}{\partial z} + K\phi = 0 \quad \text{on } z = 0 \qquad (4)$$

3) Bottom boundary condition:

$$\frac{\partial\phi}{\partial z} = 0, \quad z = d \qquad (5)$$

4) Radiation condition:

$$\lim_{R\to\infty} \sqrt{R}\left(\frac{\partial}{\partial R} - ik_0\right)\phi = 0 \qquad (6)$$

where $K = \frac{\sigma^2}{g}$, and k_0 is the finite depth wave number defined by

$$k_0 \sinh k_0 d - K \cosh k_0 d = 0 \qquad (7)$$

Moreover, the incident and diffraction potentials satisfy the body surface condition

$$\frac{\partial \phi_I}{\partial \mathbf{n}} = -\frac{\partial \phi_D}{\partial \mathbf{n}} \quad \text{on } r = a \qquad (8)$$

where $\mathbf{n}$ denotes the normal vector from body surface to fluid.

2.1 Incident Potential

The incoming waves of amplitude A and frequency σ propagating in the positive x-direction can be described by the following incident velocity potential,

$$\phi_I = \frac{Ag}{\sigma}\frac{\cosh k_0(z-d)}{\cosh k_0 d} e^{ik_0 R\cos\psi} \qquad (9)$$

Using Thorne's expansion [4], the incident potential can be expressed in terms of associated Legendre polynomial as:

$$\begin{aligned}\phi_I \;=\; & \frac{Ag}{2\sigma\cosh k_0 d}\sum_{m=0}^{\infty}\epsilon_m i^m \cos m\psi \times \\ & \sum_{s=m}^{\infty}\left\{(-1)^{s+m}e^{k_0(d-h)} + e^{k_0(h-d)}\right\} \\ & \times \frac{(k_0 r)^s}{(s+m)!}P_s^m(\cos\theta) \qquad (10)\end{aligned}$$

where $\epsilon_0 = 1$ and $\epsilon_m = 2$ for $m \geq 1$. $P_s^m(\cos\theta)$ is the associated Legendre polynomial. We can write for our convenience,

$$\phi_I(r,\theta,\psi) = \sum_{m=0}^{\infty}\hat{\phi}_I(r,\theta)\cos m\psi \qquad (11)$$

where

$$\begin{aligned}\hat{\phi}_I(r,\theta) \;=\; & \frac{Ag}{\sigma}\epsilon_m i^m \sum_{s=0}^{\infty}\chi_s \frac{(k_0 r)^{s+m}}{(s+2m)!} \\ & \times P_{s+m}^m(\cos\theta) \qquad (12)\end{aligned}$$

with

$$\begin{aligned}\chi_s \;&=\; \frac{(-1)^s e^{k_0(d-h)} + e^{-k_0(d-h)}}{2\cosh k_0 d} \\ &=\; \begin{cases}\frac{\cosh k_0(d-h)}{\cosh k_0 d}, & s = 0,2,4,6,\ldots \\ -\frac{\sinh k_0(d-h)}{\cosh k_0 d}, & s = 1,3,5,\ldots\end{cases} \qquad (13)\end{aligned}$$

2.2 Diffraction Potential

The diffraction velocity potential ϕ_D satisfies equations (3)-(6) and (8). We can express this potential by making it ψ-independent as:

$$\phi_D(r,\theta,\psi) = \sum_{m=0}^{\infty}\hat{\phi}_D(r,\theta)\cos m\psi \qquad (14)$$

where the ψ-independent potential is

$$\hat{\phi}_D(r,\theta) = \sum_{n=m}^{\infty} a^{n+2}A_{mn}G_n^m \qquad (15)$$

Here A_{mn} are the unknown complex coefficients and G_n^m are the multipole potentials. Multipole potentials are solutions of Laplace's equation which satisfy the free surface and bottom boundary conditions and behave like outgoing waves from the singular point which in this case is the centre of the sphere.

G_n^m can be expressed as

$$\begin{aligned}G_n^m \;=\; & \frac{P_n^m(\cos\theta)}{r^{n+1}} + \frac{P_n^m(\cos\alpha)}{r_1^{n+1}} + \frac{1}{(n-m)!} \\ & \int_0^{\infty}\frac{(K+k)[e^{-k(d+H)} + (-1)^{n+m}e^{-kh}]}{k\sinh kd - K\cosh kd} \\ & \times k^n \cosh k(z-d) J_m(kR)dk \qquad (16)\end{aligned}$$

The quantities α and r_1 are defined as :

$$r_1 = \sqrt{R^2 + (d+H-z)^2},\; \tan\alpha = \frac{R}{d+H-z}$$

where R, d and H have already been defined.

The potentials G_n^m and ϕ_D satisfy Laplace's equation, free surface condition, bottom surface condition and the radiation condition.

The second and third terms in equation (16) can be expanded in the region near the body surface into a series of associated Legendre's polynomials by

$$\frac{P_n^m(\cos\alpha)}{r_1^{n+1}} = \sum_{s=0}^{\infty} B_{ns}^m \left(\frac{r}{2H}\right)^{s+m} P_{s+m}^m(\cos\theta) \quad (17)$$

and

$$\int_0^\infty \frac{(K+k)[e^{-k(d+H)} + (-1)^{n+m}e^{-kh}]}{k\sinh kd - K\cosh kd} \times \frac{1}{(n-m)!} k^n \cosh k(z-d) J_m(kR)dk = \sum_{s=0}^{\infty} C_s(n,m) \left(\frac{r}{2H}\right)^{s+m} P_{s+m}^m(\cos\theta) \quad (18)$$

where B_{ns}^m and $C_s(n,m)$ are given by

$$B_{ns}^m = \frac{1}{(2H)^{n+1}} \frac{(s+n+m)!}{(s+2m)!(n-m)!} \quad (19)$$

$$C_s(n,m) = \frac{(2H)^{s+m}}{(n-m)!(s+2m)!} \times \int_0^\infty \frac{(K+k)[e^{-k(d+H)} + (-1)^{n+m}e^{-kh}]}{k\sinh kd - K\cosh kd} \times u_s(kH)dk \quad (20)$$

with $u_s(kH)$ as

$$u_s(kH) = \begin{cases} \cosh kH, & s = 0, 2, 4, \ldots \\ -\sinh kH, & s = 1, 3, 5, \ldots \end{cases} \quad (21)$$

Hence the multipole potentials G_n^m can finally be written as

$$G_n^m = \frac{P_n^m(\cos\theta)}{r^{n+1}} + \sum_{s=0}^{\infty} [B_{ns}^m + C_s(n,m)] \times \left(\frac{r}{2H}\right)^{s+m} P_{s+m}^m(\cos\theta) \quad (22)$$

From the expressions for G_n^m and $\hat{\phi}_I$ from equations (22) and (12) respectively and using the orthogonality property of associated Legendre polynomials, we arrive at

$$\sum_{n=m}^{\infty} A_{mn} E_{ns}^m = T_s^m \quad \text{for } s = m, m+1, m+2, \ldots \quad (23)$$

where

$$T_s^m = -\frac{Agk_0}{\sigma} \epsilon_m i^m (k_0 a)^{s-1} \frac{s}{(s+m)!} \chi_{s-m} \quad (24)$$

$$E_{ns}^m = -(n+1)\delta_{ns} + D_n^m(s-m) \quad (25)$$

$$D_n^m(s) = a^{n+1}(s+m) \left(\frac{a}{2H}\right)^{s+m} \times [C_s(n,m) + B_{ns}^m] \quad (26)$$

Equation (23) is a complex matrix equation in the unknowns A_{mn}. Since the infinite series appearing in equations (24) and (26) have excellent truncation property, the infinite matrices can be truncated at a certain term to solve equation (23) numerically. Commercially available complex matrix inversion routines are used to obtain the solution of the modified equation. Once these coefficients are known, the diffraction problem is completely known.

3 Problem Solution

The forces associated with the incident and diffraction potentials are the exciting forces which play a very important role in the wave field for a structure in water. The exciting forces $F_j^{(e)}$ can be obtained from:

$$F_j^{(e)} = 2i\rho a^2 \sigma A \int_0^\pi \int_0^\pi \phi|_{r=a} n_j \sin\theta d\theta d\psi \quad (27)$$

where $j = 0$ corresponds to heave motion and $j = 1$ to surge motion.

$$n_j = -P_1^j(\cos\theta)\cos j\psi, \quad j = 0, 1 \quad (28)$$

Applying the body surface condition (8) and after some simplifications, we have

$$\phi|_{r=a} = a \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \frac{2n+1}{n} A_{mn} \times P_n^m(\cos\theta)\cos m\psi \quad (29)$$

Now the exciting forces are given by

$$F_j^{(e)} = -\frac{2i\rho\sigma a^2 A\pi}{\epsilon_j} \int_0^\pi \sum_{n=j}^{\infty} a \left(\frac{2n+1}{n}\right) \times A_{jn} P_n^j(\cos\theta) \sin\theta d\theta \quad (30)$$

where $\epsilon_j = 1$ for $j = 0$, $\epsilon_j = 2$ for $j \geq 1$.

Using the orthogonality property of associated Legendre polynomials, we obtain

$$F_j^{(e)} = -4i\rho\sigma\pi a^3 A A_{j1} \qquad (31)$$

Hence the surge exciting force $F_x^{(e)} = f_{xd}$ and the heave exciting force $F_z^{(e)} = f_{zd}$ are given by

$$\frac{f_{xd}}{4i\rho\sigma A\pi a^3} = -A_{11} \qquad (32)$$

and

$$\frac{f_{zd}}{4i\rho\sigma A\pi a^3} = -A_{01} \qquad (33)$$

After non-dimensionalizing the expressions in equations (32) and (33) we present the results of the analytical expressions for the exciting forces due to surge and heave motions. The complex matrix equation (23) is to be solved in order to determine the unknown coefficients A_{mn} for $m = 0$ and $m = 1$. To compute the horizontal exciting force, f_{xd}, we need to solve the equation (32) and the vertical exciting force, f_{zd}, is evaluated by solving equation (33). This infinite system of equations represented by equation (23) is made finite by truncating as

$$\sum_{n=0}^{N_p} A_{mn} E_{ns}^m = T_s^m \qquad (34)$$

Tables 1 to 4 give us the exciting force coefficients for both fixed submergence and fixed depth. The results have been compared with the results of Wang [3] and Wu *et al.* [7] and they seem to agree with those sets of results. The first two tables present the surge and heave exciting forces for a fixed submergence $h/a = 1.25$ for various depths, e.g. $d/a = 2.5$, 5, 11.0 and $d/a = 20.0$. Table 3 and Table 4 present us the surge and heave exciting forces for a fixed depth $d/a = 6.0$ but for a set of different submergence values.

	←		d/a	→
Ka	2.5	5.0	11.0	20.0
.10	3.1539	2.1872	1.5893	1.4897
.20	2.1152	1.5902	1.3151	1.2621
.30	1.6347	1.1861	1.1361	1.1102
.40	1.2862	0.9861	0.9858	0.9826
.50	1.1134	0.8862	0.8852	0.8834
.60	0.9217	0.8682	0.7809	0.8124
.70	0.7692	0.7398	0.6947	0.7395
.80	0.6824	0.6482	0.6345	0.6315
.90	0.5824	0.5789	0.5786	0.5785
1.00	0.5037	0.4925	0.4911	0.4901
1.20	0.3476	0.3391	0.3379	0.3377

Table 1: Surge exciting force (h/a=1.25)

4 Conclusion

The work is motivated by the need for analytical solutions for the exciting forces. It has been shown that the body submergence and depth have influence on the exciting forces. The use of associated Legendre polynomial reduces the solutions to simpler forms. The pitch motion was not considered since, for a spherical body, the moment acting on it becomes zero. The radiation problem can be considered in a similar manner and that will help us know about added-mass and damping effect of the structure. It will be interesting to extend the investigation further to consider two or more submerged spheres. The analysis of interaction among several structures nearer to each other would be more important from the practical point of view. The main challenge will be the problems related to arbitrarily shaped geometries for which the above work will contribute only qualitatively.

References:

[1] T.H. Havelock, The Wave Resistance of a Spheroid, *Proc. Royal Society, London*, Vol.A131, 1931, pp. 275-284.

[2] A Hulme, The Wave Forces Acting on a Floating Hemisphere Undergoing Forced Periodic Oscillations, *J. Fluid Mech.*, No.121, 1982 pp. 443-463.

[3] S. Wang, Motions of a Spherical Submarine in Waves, *Ocean Engg.*, No.13, 1986, pp. 249-

	←		d/a	→
Ka	2.5	5.0	11.0	20.0
.1	0.8241	1.2041	1.3979	1.4671
.20	0.7965	1.1505	1.3192	1.3294
.30	0.7752	1.1421	1.2547	1.2609
.40	0.7598	1.1167	1.1147	1.1162
.50	0.7421	0.9917	0.9867	0.9872
.60	0.7134	0.9256	0.9269	0.9283
.70	0.6790	0.8291	0.8304	0.8317
.80	0.6224	0.7398	0.7404	0.7409
.90	0.5631	0.6123	0.6136	0.6149
1.00	0.4832	0.4846	0.4850	0.4852
1.10	0.4162	0.4247	0.4261	0.4275
1.20	0.3281	0.3286	0.3284	0.3283

Table 2: Heave exciting force (h/a=1.25)

	←	h/a	→
Ka	1.25	1.75	3.00
.1	2.0117	1.8694	1.7021
.2	1.5106	1.2864	0.9462
.3	1.2461	0.9862	0.6741
.4	1.0967	0.7421	0.3909
.5	0.8984	0.6842	0.3646
.6	0.7791	0.5098	0.2517
.7	0.7364	0.4726	0.2021
.8	0.6274	0.3622	0.1271
.9	0.5097	0.2671	0.0983
1.0	0.4892	0.2491	0.0608
1.2	0.3972	0.1977	0.0323
1.4	0.2947	0.1389	0.0086
1.6	0.2566	0.1082	0.0016
1.8	0.2314	0.0627	0.0009

Table 3: Surge exciting force (d/a=6)

	←	h/a	→
Ka	1.25	1.75	3.00
.1	1.2561	1.0692	0.6841
.2	1.2293	0.9542	0.6194
.3	1.2007	0.7781	0.4382
.4	1.1467	0.7392	0.3922
.5	0.9724	0.6107	0.2965
.6	0.8862	0.5566	0.2264
.7	0.6833	0.4192	0.1791
.8	0.6374	0.3643	0.1267
.9	0.5277	0.2818	0.1082
1.0	0.4721	0.2364	0.0927
1.4	0.2021	0.1028	0.0237
1.8	0.1161	0.0711	0.0081
2.0	0.0986	0.0529	0.0072
2.4	0.0583	0.0294	0.0039
2.8	0.0185	0.0129	0.0014
3.0	0.0011	0.0081	0.0005

Table 4: Heave exciting force (d/a=6)

271.
[4] R.C. Thorne, Multipole Expansions in the Theory of Surface Waves, *Proc. Camb. Phil. Soc.*, Vol.49, 1953, pp. 707-716.
[5] G.X. Wu and R. Eatock Taylor, Radiation and Diffraction of Water Waves by a Submerged Sphere at Forward Speed, *Proc. Royal Soc., London*, Vol.A417, 1988, pp. 433-461.
[6] G.X. Wu and R. Eatock Taylor, The Hydrodynamic Forces on a Submerged Sphere Moving in a Circular Path. *Proc. Royal Society, London*, Vol.A428, 1990, pp. 215-227.
[7] G.X. Wu, J.A. Witz, Q. Ma and D.T. Brown, Analysis of Wave Induced Forces Acting on a Submerged Sphere in Finite Water Depth, *Appl. Ocean Research*, Vol16, 1994, pp. 353-361.
[8] S.N. Bora, *The Interaction of Water Waves with Submerged Spheres and Circular Cylinders*, PhD thesis, Department of Applied Mathematics, Technical University of Nova Scotia, Canada, 1998.