Alternating cycles in soliton graphs

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Abstract: Soliton graphs having an alternating cycle are characterized with the help of a shrinking procedure. This characterization leads to a method testing the existence of an alternating cycle in a soliton graph. The suggested algorithm runs in $\mathcal{O}(n^3)$ time, where n is the number of vertices in the graph.

Key-Words: Combinatorial Problems, Graph Matchings, Alternating cycles, Soliton graphs, 1-extendable graphs

1 Introduction

One of the most ambitious goals of research in modern bioelectronics is to develop a molecular computer. Inspired by this research soliton automata were introduced in [4] to serve as a mathematical model for certain molecular switching devices. Many interesting special cases of soliton automata have been described (see e.g [5]), but it was not until [1] that matching theory was recognized as the fundamental theoretical background for the study of this model.

The underlying object of a soliton automaton is a so called soliton graph. Such a graph is the topological model of a hydrocarbon molecule chain. In order for the graph to act as an automaton we need to de£ne its states. To reach this goal we use certain matchings, called perfect internal matchings, of the graph, where by a *matching* of graph G we mean a set of edges, without two incident ones to the same vertex. A soliton graph must have a *perfect internal matching*, which is a matching covering all vertices with degree at least 2. These vertices – called *internal* – model carbon atoms, whereas vertices with degree one – called *external* – represent a suitable chemical interface with the outside world. Because of the chemical background, the name *state* is also used as a synonym for perfect internal matching. In addition to possessing a state, a *soliton graph* is also expected to have an external vertex. The edges of a soliton graph G are also distinguished between, such as *allowed* (contained in some state of G) or *forbidden* (not contained in any state of G).

The analysis of soliton automata is a complex task, and the general case is still open. Therefore it is a central problem to describe the structure of soliton graphs with respect to their states. In [2] a decomposition of soliton graphs into elementary components – maximal connected subgraphs spanned by allowed edges only – was worked out, and these components were grouped into pairwise disjoint families based on how they can be reached by alternating paths starting from external vertices. From a practical point of view the most important special case is the class of deterministic soliton automata, and consequently the graphs associated with them, called deterministic soliton graphs. The graph-theoretic characterization of deterministic soliton graphs is given in [3], where it was proved that a soliton graph G is deterministic iff it does not contain an alternating cycle with respect to any state of G. However, this characterization does not provide a direct method to solve the important practical problem of checking the determinism of a soliton graph ef£ciently, as a graph might have an exponential number of states.

In this paper we show that testing the existence of an alternating cycle, and thus testing the determinism, can be solved in $\mathcal{O}(n^3)$ time, where n denotes the number of vertices. To reach this goal £rst we show in Section 3.1 that the general problem can be reduced to *1-extendable graphs*, which are connected graphs without forbidden edges. Then in Section 3.2 an ear decomposition of soliton graphs is worked out to serve as a technique for their structural description. The main result is based on the shrinking operation presented in Section 3.3, by which certain internal vertices with degree 2 are eliminated. With the help of the shrinking operation we obtain a characterization of soliton graphs with alternating cycles, which directly leads to an ef£cient algorithm.

2 Problem Formulation

By a graph, throughout the paper, we mean a £nite undirected graph in the most general sense, with multiple edges and loops allowed. Our notation and terminology will be compatible with that of [6], except for the words "point" and "line" being replaced by "vertex" and "edge", respectively.

Let G be a graph with set of vertices $V(G)$ and set of edges $E(G)$. The sets of external and internal vertices of G will be denoted by $Ext(G)$ and $Int(G)$ respectively, while for the degree of a vertex v the notation $d(v)$ will be used. Graph G is called *open* if $Ext(G) \neq \emptyset$, otherwise G is *closed*. *External edges* are those that are incident with at least one external vertex, and an *internal edge* is one that is not external. For a subgraph G' and matching M of G, $M_{(G')}$ will denote the restriction of M to G'. A subgraph G' of G is *nice* if it has a perfect internal matching and every perfect internal matching of G' can be extended to a perfect internal matching of G. A matching is called *perfect* if it covers all vertices of G.

Let G be a soliton graph, £xed for the rest of this section, and let M be a state of G. An edge $e \in E(G)$ is said to be *M-positive* (*M-negative*) if $e \in M$ (respectively, $e \notin M$). An *M-alternating path (cycle)* in G is a path (respectively, even-length cycle) stepping on M-positive and M-negative edges in an alternating fashion. Let us agree that, if the matching M is understood or irrelevant in a particular context, then it will not explicitly be indicated in these terms.

An *external alternating path* is one that has an external endpoint, whereas an *internal alternating path* is one that is not external. An alternating path is *positive* (*negative*) if it is such at its internal endpoints, meaning that the edges incident with those endpoints are positive (respectively, negative). If both endpoints of an alternating path are external, then it is called a *crossing*. For a path α and vertices $u, v \in V(\alpha)$ we will use the notation $\alpha[u, v]$ by which we mean the subpath of α having endpoints u and v.

An *alternating unit* is either a crossing or an alternating cycle. *Switching* on a positive alternating path or alternating unit α amounts to changing the sign of each edge along the path (respectively, unit). It is easy to see that the operation of switching on α creates a new matching $S(M, \alpha)$ for G, which matching is a state if the switching is carried out on an alternating unit. The following two results from [1] characterize 1-extendable graphs by alternating units.

Proposition 1 *Any edge of a 1-extendable graph G is traversed by an alternating unit in every state of G.*

Proposition 2 *A soliton graph G is 1-extendable iff for any two edges $e, f \in E(G)$ there exists an alternating unit containing both e and f.*

The basic aim of the paper is to test if a soliton graph contains an alternating cycle. The motivation of the problem solving is the practical question of deciding if a soliton graph is deterministic. In [3] it was proved that the above two problems are equivalent. The naive approach, i.e. listing all the states of the graph and checking the existence of an alternating cycle in each state, is inef£cient, as a soliton graph might have exponential number of states. So far only the following suf£cient condition was proved (see [3]).

Proposition 3 *Let G be a 1-extendable soliton graph and G' is a 1-extendable nice subgraph of G such that $G = G' + e_1 + \ldots + e_k$ with $k \geq 2$, where $e_1, \ldots, e_k$ are internal edges in G. If for any subset $\{e_{i_1}, \ldots, e_{i_r}\} \subset \{e_1, \ldots, e_k\}$ with $1 \leq r \leq k-1$, $G' + e_{i_1} + \ldots + e_{i_r}$ is not 1-extendable, then G contains an alternating cycle.*

In the followings we will characterize soliton graphs with alternating cycles, which results in an ef£cient method.

3 Problem Solution

We will solve the problem through three steps described in Sections 3.1-3.3. As an alternating cycle can contain only allowed edges, we present in Section 3.1 the closure operation on open graphs and show how it can be applied for identifying the allowed edges. This algorithm results in a decomposition of allowed edges into maximal 1-extendable subgraphs. Then by making use of the ear-decomposition described in Section 3.2, a reduction procedure is worked out in Section 3.3 proving that our problem can be simpli£ed to searching even length cycles in reduced 1-extendable soliton graphs.

3.1 The closure of open graphs

In an open graph G, connect the external vertices of G to each other in all possible ways. Furthermore, if $|V(G)|$ is odd, then add a new vertex c and introduce edges from c to all the external vertices. The resulting graph G^* is called the *closure* of G. It is easy to see, cf. [2], that G has a perfect internal matching iff G^* has a perfect matching. Clearly, the perfect internal matchings of G can be obtained from the perfect matchings of G^* simply by restricting these matchings to $E(G)$. Furthermore, it was also proved in [2] that an edge of graph G is allowed in G iff it is such in G^*. Making use of the above observations, a perfect internal matching and the allowed edges in an open graph can be determined by the following way.

Consider the closure G^* of a given open graph G. Using the Edmonds algorithm (see [6]) determine a perfect matching of G^*, if there exists one. If the output becomes that G is a soliton graph, then by using the algorithm of [7], identify the allowed edges in G^*. It can be proved (see [7]) that the complexity of the procedure for identifying the allowed edges is the same as that of the Edmonds algorithm. The Edmonds algorithm can be implemented in $\mathcal{O}(n*m)$ time – where m denotes the number of edges in G^* and n denotes the number of vertices in G^* –, but the closure operation may result in $\mathcal{O}(n^2)$ new edges, thus the complexity of the above method is $\mathcal{O}(n^3)$.

It is clear that a forbidden edge is not contained in any alternating cycle. (If an edge e is traversed by an alternating cycle α in state M, then either M or state $S(M,\alpha)$ contains e.) Therefore, by obtaining the output of the above algorithm the forbidden edges can be removed from G. Then the connected components of the resulted graph are 1-extendable. If such a component G_i does not contain external vertices of G, then, according to Proposition 1, the existence of an alternating cycle in G_i is equivalent to the condition that G_i does not consist of a single edge. Consequently, in the rest of the paper we need to concentrate only on the 1-extendable components containing external vertices. Therefore, in order to solve the general problem, it is enough to £nd an ef£cient method to decide if an open 1-extendable graph possesses an alternating cycle.

3.2 Ear decomposition of 1-extendable soliton graphs

Ear decomposition was recognized as a useful technique for studying the structure of 1-extendable graphs with respect to perfect matchings (see [6]). In this section we present an ear structure of 1-extendable soliton graphs without alternating cycles. Theorem 4 will play an important role in proving the main result of the paper.

Theorem 4 *Let G be a 1-extendable soliton graph without alternating cycles. Then there exists a sequence $(G_0, G_1, \ldots, G_t)$ of nice 1-extendable subgraphs of G with the following properties:*

(i) G_0 is a tree containing all of G's external vertices;

(ii) For each k, $1 \leq k \leq t$, G_k arises from G_{k-1} by adding one odd length path, called ear, having both endpoints – but no other points – in G_{k-1};

(iii) $G_t = G$.

Proof. First we prove that there exists a graph G_0 with the conditions of (i). To this end let v be an external vertex of G and consider a maximal nice subgraph T of G with the property P that T is a tree containing v such that $Ext(T) \subseteq Ext(G)$. According to Proposition 1, v is traversed by an alternating crossing γ. Therefore γ has property P, so a suitable T exists. Now suppose on the contrary that $Ext(T) \neq Ext(G)$ and let G_1 denote a maximal nice 1-extendable subgraph of G with the property P_1 meaning $Ext(G_1) = Ext(T)$. Furthermore, let M denote a state of G such that $M_{(G_1)}$ is a perfect internal matching in T. Based on our assumption $G_1 \neq G$, so there exists a vertex $w \in V(G) \setminus V(G_1)$ that is adjacent to some vertex $u \in V(G_1)$. G does not contain alternating cycle, thus, according to Proposition 1, there exists an M-alternating crossing α in G containing the edge (u,w). We claim that one endpoint x of α belongs to $Ext(G) \setminus Ext(G_1)$. Indeed, in other cases $G_1 + \alpha$ would constitute a subgraph of G satisfying property P_1, which contradicts the fact that G_1 is maximal. Now starting from x let y denote the £rst vertex of α for which $y \in V(G_1)$. It easily follows from Proposition 1, that G_1 contains a positive external M_{G_1}-alternating path α' leading to y. Therefore $\alpha[x,y] + \alpha'$ forms a crossing β in G, consequently β has a pre£x β_1 connecting x with an internal vertex of T such that β_1 is edge-disjoint from T. However $T + \beta_1$ has property P contradicting the maximality of T. Thus we conclude that $Ext(T) = Ext(G)$, by which T is a suitable choice for G_0.

Now for the proof of (ii) and (iii), as an induction hypothesis we can suppose that an appropriate 1-extendable nice subgraph G_k of G with $k \geq 0$ is given. If G=G_k, then we are ready. So suppose there is an edge e of $G - G_k$ with at least one endpoint in G_k. Let M be a state of G such that $M_{(G_k)}$ is a perfect internal matching in G_k, and let α be an M-alternating crossing passing through e. Then it is clear that the edges of $E(\alpha) \setminus E(G_k)$ constitute a set of ears $\mathcal{P} = \{\beta_1, \ldots, \beta_m\}$, $m \geq 1$, with respect to G_k. Therefore we only need to prove that for some $1 \leq j \leq m$, $G_k + \beta_j$ is 1-extendable.

Clearly, we may assume without the loss of generality that each ear of $\mathcal{P}$ is a single edge. Then, by using an induction on m, we can easily £nish the proof with the help of Proposition 3. Indeed, if $m = 1$, then there is nothing to prove. In other cases, according to Proposition 3, there exists a subset $\mathcal{P}' = \{\beta_{i_1}, \ldots, \beta_{i_l}\}$ of $\mathcal{P}$ $(l < m)$, such that $G_k + \beta_{i_1} + \ldots + \beta_{i_l}$ is 1-

extendable. Then, according to the induction hypothesis, there exists an element β_{i_j} of $\mathcal{P}'$ with $G_k + \beta_{i_j}$ being 1-extendable. Therefore the proof is complete. $\diamondsuit$

3.3 Minimal representation of 1-extendable soliton graphs

In this section we will show how to reduce the problem of searching alternating cycles to checking the existence of even cycles. A reduction procedure must be introduced for this goal, which is based on the shrinking construction described in Proposition 5.

Proposition 5 *Let $e_1 = (v_1, v)$ and $e_2 = (v, v_2)$ be internal edges of graph G such that v, v_1 and v_2 are distinct vertices and the degree of v is 2. Moreover, let $E_1 = \{e \in E(G) \mid e \neq e_1$ and e is incident with v_1 in G $\}$ and let E' denote the set of edges obtained from the elements of E_1 by replacing v_1 with v_2 at the appropriate endpoint of each edge of E_1.*
Then $G' = G - \{v, v_1\} + E'$ is a soliton graph iff G is a soliton graph.

Proof. Straightforward, omitted. $\diamondsuit$

Intuitively, G' is obtained from G by shrinking the path $\alpha = v_1, e_1, v, e_2, v_2$ to a single vertex. The *minimal representation* $\mathcal{R}(G)$ of graph G is the graph obtained by repeated applications of the above construction to G until no further shrinking action can be done. A graph is called *reduced* if $G = \mathcal{R}(G)$.

Proposition 6 *A soliton graph G contains an alternating cycle iff $\mathcal{R}(G)$ does.*

Proof. It is enough to show that one shrinking action and its inverse preserve alternating cycles. To this end £rst let M be a state of G, β be an M-alternating cycle in G and $(v_1, v), (v, v_2)$ be the edges through which the shrinking action is applied. Then either $(v_1, v) \in M$ or $(v, v_2) \in M$. Consequently, either $(v_1, v), (v, v_2) \in E(\beta)$ or at most one of v_1 and v_2 is contained in β. In both cases the claim of the 'Only if' part is straightforward.

Conversely, it is also clear that an alternating cycle α of $\mathcal{R}(G)$ is also present in G, if no vertex of α is the result of the shrinking. Otherwise, if a shrinking has resulted in a vertex v of α, then we evidently obtain, after blowing-up through v, that the extended α is also alternating. $\diamondsuit$

Proposition 7 *The minimal representation of a 1-extendable graph is also 1-extendable.*

Proof. Let G be a 1-extendable graph, v_1, v_2 and v be distinct internal vertices with $d(v) = 2$ and $(v_1, v), (v, v_2) \in E(G)$. Apply a shrinking action through vertex v and let G' denote the resulted graph. It is enough to prove that any edge e of G' is allowed.

If at most one endpoint of e is incident with the shrunken vertex, then we can prove, by using the same argument as in the proof of the 'Only if' part of Proposition 6, that the alternating unit containing e in G is preserved after the shrinking. Therefore, if we show that e is not a looping edge around the shrunken vertex, then we are ready. The above statement is indeed true, as otherwise either e is a looping edge in G or it connects v_1 and v_2, in both cases meaning that e is forbidden in G; which is a contradiction. $\diamondsuit$

A connected graph is a *generalized tree* if it does not contain even-length cycles. Note that any edge of a generalized tree G is contained in at most one cycle of G. In the proof of the main result we will make use of the following lemma.

Lemma 8 *Let G be a generalized tree with $u, v \in Int(G)$ such that $G + (u, v)$ is a reduced 1-extendable graph. Then u and v are connected by a positive alternating path with respect to some state of G iff there exists an odd length path between u and v in G.*

Proof. We may suppose without the loss of generality that G is a reduced 1-extendable generalized tree. Indeed, let us cut $G + (u, v)$ at the edge $e = (u, v)$, that is, replace (u, v) by two new external edges, each being incident with one internal endpoint of e. Then it is easy to see that the resulted graph is a 1-extendable generalized tree, and it contains a positive alternating path between u and v iff there exists such a path in G.

Let α be an odd-length path connecting u and v and let M^α be the unique perfect matching of α. Moreover, let $G_1, \ldots, G_k$ denote the connected components of $G - E(\alpha)$. It is easy to see – as G is a generalized tree – that for each $1 \leq i \leq k$, $1 \leq \mid V(G_i) \cap V(\alpha) \mid \leq 2$. Now construct a graph G'_i from G_i for $1 \leq i \leq k$ in the following way: If $\mid V(G_i) \cap V(\alpha) \mid = 1$, then let $G'_i = G_i$, otherwise extend G_i by the subpath of α connecting the vertices v_i^1, v_i^2 of $V(G_i) \cap V(\alpha)$ and attach a new external edge to v_i^j ($j = 1, 2$). Note that the subgraphs $G'_1, \ldots, G'_k$ are pairwise edge-disjoint. We will construct a state M^i for each G'_i such that $M^\alpha_{(\alpha_i)} = M^i_{(\alpha_i)}$ for the subpath α_i constituted by $E(\alpha) \cap E(G'_i)$. By the above fact our proof will be complete, as for any state M of G corresponding to the union of the matchings with the above conditions, $M_{(\alpha)} = M^\alpha$ will hold.

Suppose £rst that G_i has a unique common vertex with α. Then a state M' of G exists such that $M'_{(G_i)}$ is a perfect internal matching in G_i (remember that G

is 1-extendable), thus in this case $M'_{(G_i)}$ is a suitable choice for M^i.

Now consider the case when $| V(G_i) \cap V(\alpha) |= 2$. Then the subpath α_i of α belonging to G'_i is a part of an odd-length cycle β_i of G'_i. Let $w_i^1, \ldots, w_i^l$ denote the vertices of β_i. Then for each $1 \leq j \leq l$, the notation G_i^j will be used for the maximal connected subgraph of G_i with w_i^j being the unique common vertex of G_i^j and β_i. Note that the subgraphs $G_i^1, \ldots, G_i^l$ are pairwise vertex-disjoint. Observe that for any $1 \leq j \leq l$, G_i^j has a perfect internal matching M_{ij}^p and a matching M_{ij}^n covering all internal vertices but w_i^j. Indeed, it is clear that G'_i is a reduced 1-extendable generalized tree, thus there exist states M^{i1} and M^{i2} of G'_i such that $M^{i1}_{(G_i^j)}$ is a perfect internal matching of G_i^j and $M^{i2}_{(G_i^j)}$ is a matching covering all vertices but w_i^j in G_i^j. Now let $M_i^n = \cup_{j=1}^{l} M_{ij}^n$ and construct a maximum matching M^{β_i} of β_i such that the restriction of M^{β_i} to α is equal to the set of edges of α_i covered by M^α. It is clear that M^{β_i} exists with the property of possessing a unique uncovered vertex w_i^r of β_i. Therefore the state $M^{\beta_i} \cup M_{ir}^p \cup (M_i^n \setminus M_{ir}^n)$ is a suitable choice for M^i, by which the proof is complete. $\diamondsuit$

Now we are ready to prove our main result.

Theorem 9 *A 1-extendable soliton graph G contains an alternating cycle iff $\mathcal{R}(G)$ is not a generalized tree.*

Proof. If $\mathcal{R}(G)$ is a generalized tree, then according to Proposition 6, G does not contain an alternating cycle, thus it is enough to prove the 'If' part. We will use induction on the number of internal edges of $\mathcal{R}(G)$.

The basis steps are trivial, because a generalized tree with at most one internal edge is clearly a tree. For the induction step suppose on the contrary that G is a 1-extendable soliton graph without alternating cycle, but $\mathcal{R}(G)$ contains an even-length cycle α. We will show that the above condition implies the existence of an alternating cycle in $\mathcal{R}(G)$, which is a contradiction because of Proposition 6.

We know by Propositions 6 and 7 that with our assumption $\mathcal{R}(G)$ is a 1-extendable soliton graph which does not contain alternating cycles, thus we can apply Theorem 4 for $\mathcal{R}(G)$. Note that the last ear in the above ear-decomposition is a single edge e, as $\mathcal{R}(G)$ is a reduced graph. Now let v and w denote the endpoints of e and let $G' = \mathcal{R}(G) - e$.

If G' is a generalized tree, then α must contain e. Consequently v and w are connected by an odd length path in G', which implies, by applying Lemma 8, that there exists an alternating cycle traversing e.

Otherwise suppose that $\mathcal{R}(G')$ is not a generalized tree. In this case the existence of an alternating cycle in G' follows from the induction hypothesis and from Proposition 6.

As a consequence of the preceding two paragraphs, we may assume for the rest of the proof that $\mathcal{R}(G')$ is a generalized tree, but G' contains an even-length cycle. Therefore at least one of v and w, say vertex v, has degree 2 in G' such that it is adjacent to distinct internal vertices. Apply a shrinking action in G' through vertex v, and let G^v denote the resulted graph. There are two cases.

Case 1. G^v is a generalized tree. In this case let v_1 and v_2 denote the vertices adjacent to v in G'. Then it is clear that both v_1 and v_2 must be present in each even cycle of G', but no even cycle may contain v. Based on this observation, we can describe the structure of G'.

The even-length cycles of G', denoted by $\beta_1, \ldots, \beta_m$, are pairwise edge-disjoint and $V(\beta_i) \cap V(\beta_j) = \{v_1, v_2\}$ for every $1 \leq i, j \leq m$. Furthermore, for any vertex x of $G_\beta = \beta_1 \cup \ldots \cup \beta_m$, there exists a maximal connected subgraph G_x of G' such that x is the unique common vertex of G_x and G_β. Note that for any two distinct $x, y \in V(G_\beta)$, G_x and G_y are vertex-disjoint, and $G_x \neq x$ for any $x \in V(G_\beta)$ being different from v_1, v_2 and w. Now we can conclude that if $V(G_\beta) = \{x_1, \ldots, x_k\}$, then $G' = G_\beta + G_{x_1} + \ldots + G_{x_k} + (v_1, v) + (v, v_2)$.

G' is 1-extendable, thus as we have seen in a similar situation in the proof of Lemma 8, for any $x \in V(G_\beta)$, there exists a matching M_x^n of G_x covering all internal vertices but x of $V(G_x)$, and G_x has a perfect internal matching M_x^p. We will use the above notations throughout the proof.

Now let x denote the vertex of $V(G_\beta)$ for which $w \in G_x$. Moreover, let β_i denote the cycle containing x and let f be an edge of β_i incident with x. According to Proposition 2, there exists an M'-alternating crossing γ in some state M' of G' traversing both f and some edge incident with w. (If $w = x$, then any alternating crossing traversing f is suitable.) We may also suppose that $f \in M'$, because if the case was not this, then $S(M', \gamma)$ can be considered.

Assume £rst that x and w are at even distance on γ. Then $\gamma[x, w]$ is negative at vertex x. Let β'_i denote the subpath of β_i between v_1 and v_2 which contains x (if $x = v_1$ or $x = v_2$, then both subpaths are suitable) and let $\beta''_i = \beta_i - \beta'_i$. It is clear that the length of both β'_i and β''_i is odd. We may suppose without the loss of generality that x is at an odd distance from v_1 on β'_i and let y denote the £rst vertex after x on the subpath $\beta_i'[x, v_2]$. Let M_i be a matching of β_i covering all vertices except for v_2 and y. Now assemble the state M of $\mathcal{R}(G)$ by the following way. M consists of $M_i \cup \{(v_2, v)\} \cup M'_{(G_x)} \cup M_y^p$, a perfect matching M_j of each β_j ($j \neq i$ and $1 \leq j \leq m$), M_u^n for each element $u \in V(G_\beta)$ different from both x and y. Now the M-alternating cycle is obtained by $\beta_i'' + (v_2, v) + (v, w) + \gamma[w, x] + \beta_i'[x, v_1]$.

If x and w are at odd distance on γ, then let M_j denote a perfect matching of β_j for each $1 \leq j \leq m$ and let M_w denote the matching $S(M'_{(G_x)}, \gamma[z, w])$, where z denotes the external endpoint of γ in G_x. Then the required state M of $\mathcal{R}(G)$ consists of $M_1 \cup \ldots \cup M_m \cup M_w$, the edge (v, w), and M_u^n for each element u of $V(G_\beta)$. In this case we obtain that β_1 is an M-alternating cycle, which makes the proof of Case 1 complete.

Case 2. G^v is not a generalized tree. In this case $d(w) = 2$ in G' and w is adjacent to distinct internal vertices w_1 and w_2, as $\mathcal{R}(G')$ is a generalized tree by assumption. Construct then graph G^w analogously to G^v. If G^w is a generalized tree, then the Theorem is proved by Case 1, because in this situation the role of v and w is symmetric. Otherwise, using the same notation as in Case 1, $G^w = G_\beta + G_{x_1} + \ldots + G_{x_k} + (v_1, v) + (v, v_2)$. Observe now that w is shrunk into either v_1 or v_2 in such a way that w_1 is incident with v. Indeed, in any other case it would be easy to check that G^v would be a generalized tree. Then we may assume without the loss of generality that w is shrunk into v_2, by which $G' = G_\beta + G_{x_1} + \ldots + G_{x_k} + G_{w_1} + \beta'$, where β' consists of the edges (v_1, v), (v, w_1), (w_1, w) and (w, w_2) such that $w_2 = v_2$, and G_{w_1} is a maximal connected subgraph of G' with $V(G_{w_1}) \cap V(\beta') = \{w_1\}$. Note that $G_{w_1} \neq w_1$ and it is vertex-disjoint from $G_{x_1} + \ldots + G_{x_k}$. Then, as we have seen in the last paragraph of the proof of Case 1, a perfect internal matching M' of $G' - \beta'$ can be assembled such that G_β contains an M'-alternating cycle γ . Now $M' \cup \{(v, w)\}$, together with a perfect internal matching of G_{w_1}, constitute the required state of $\mathcal{R}(G)$ with respect to which γ is alternating .

As all possibilities were considered, and in each case we concluded that the assumption for the existence of an even-length cycle enabled us to construct an alternating cycle, the proof is complete. $\diamond$

By making use of the above result we can give an ef£cient method to decide if a 1-extendable soliton graph contains an alternating cycle, which consists of the shrinking procedure and testing the existence of an even-length cycle. The method of Section 3.1 reduced our general problem to 1-extendable soliton graphs with a time complexity of $\mathcal{O}(n^3)$, where n denotes the number of vertices. It is clear that both the shrinking procedure and testing the existence of an even-length cycle can be implemented in $\mathcal{O}(m)$ time, where m is the number of edges. Therefore the complexity of the whole problem is also $\mathcal{O}(n^3)$.

4 Conclusion

In this paper we showed that soliton graphs with alternating cycles can be characterized with the help of a shrinking operation eliminating the vertices which are only adjacent to distinct internal vertices and have degree 2. Before proving the main result £rst we presented the closure operation with the help of which classical matching algorithms were applied for identifying the forbidden edges. This observation reduced our problem to searching alternating cycles in 1-extendable soliton graphs. We proved as the main result that a 1-extendable soliton graph contains an alternating cycle if its minimal representation with respect to the shrinking operation contains an even-length cycle. This theorem suggests a method consisting of the shrinking procedure and testing the existence of even-length cycles. This method, together with the procedure identifying the forbidden edges, results in an algorithm searching alternating cycles. Our algorithm runs in $\mathcal{O}(n^3)$ time, where n is the number of vertices in the graph.

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