On Explicite Formulae for Hankel Matrix Inversion

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Abstract: In this paper new explicite formulae for Hankel matrix inversion are suggested. They are derived by the construction of special, so called fundamental Hankel or Toeplitz equations systems. The formulae differ from each other via their form, complexity and different extra assumptions on the corresponding matrix.

Key-Words: Hankel and Toeplitz matrices, Toeplitz matrix inversion, Hankel matrix inversion

1 Introduction

Hankel and Toeplitz matrices represent a specific class of matrices which have the same elements in direction of the secondary, or the main matrix diagonal, respectively. Such matrices occur in a large variety of areas in pure and applied mathematics. They often appear as discretization of differential and integral equations, they arise in physical data processing, in theory of orthogonal polynomials, stationary processes, moment problems, functional and harmonic analysis, and many others. The research on finite Toeplitz and Hankel matrices is motivated also by important applications of the Wiener-Hopf theory, e.g. in theoretical physics. Such matrices correspond to integral equations with kernels which depend on differences, or sums of arguments, respectively.

Hankel and Toeplitz matrices attract the attention by significant characteristic properties which result in a big amount of papers published on the topic devoted for various aspects of the research (see e.g. books [1, 2, 15, 18, 11, 12, 14, 3]).

The problem of inversion of a finite Hankel matrix has been studied in the connection with intensive investigation of effective methods for the solution of linear algebraic equations system with a Toeplitz matrix and the inversion of a finite Toeplitz matrix. It is a quite simple fact known from linear algebra that the inversion of a regular nxn matrix A is equivalent to the solution of n equations $Ax_k = e_k$, $k = 0, 1, ..., n-1$. It is clear that any system of less than n equations cannot determine completely the inversion of A, in general. But if the matrix A is Toeplitz or Hankel it has only $2n-1$ degrees of freedom and the inversion is already determined by a system of less than n equations, in the optimal case by two equations only. For the inversion of a finite Toeplitz matrix, if it is invertible, several explicite formulae have been derived by the solution of different, so called fundamental systems of equations [12, 11].

The problem of inversion of a finite Hankel matrix, if it is invertible, was solved by investigation of close connection between Toeplitz and Hankel matrices. Analogically to the Toeplitz case, it has been shown that it is possible to suggest explicite formulae for the inversion of Hankel matrices based on the solution of fundamental equations systems which are constructed by corresponding Hankel (reviewed in [11]), or Toeplitz (suggested in [12]) matrices. These formulae differ from each other via their form, complexity and different extra assumptions on corresponding matrix. In this paper[1] we review known formulae, derive new ones, as well as give the correct formulae which have been published in incorrect form.

2 Hankel Matrix Inversion

Let us denote vectors

$$e_0 = (1, 0, ..., 0)^T, e_1 = (0, 1, 0, ..., 0)^T, ...,$$

$$e_{n-1} = (0, ..., 0, 1)^T,$$

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$$x=(x_0,x_1,...,x_{n-1})^T, y=(y_0,y_1,...,y_{n-1})^T,$$
$$z=(z_0,z_1,...,z_{n-1})^T, w=(w_0,w_1,...,w_{n-1})^T,$$
$$u=(u_0,u_1,...,u_{n-1})^T, v=(v_0,v_1,...,v_{n-1})^T,$$
$$h(\alpha)=(a_n,a_{n+1},...,a_{2n-2},\alpha)^T,$$
$$g(\alpha)=(\alpha,a_{2n-2},a_{2n-3},...,a_n)^T,$$
$$g(\alpha)=(\alpha,a_{2n-2},a_{2n-3},...,a_n)^T,$$
$$k(\alpha)=(\alpha,a_0,a_1,...,a_{n-2})^T,$$
$$g(\alpha)=(\alpha,a_{2n-2},...,a_n)^T,$$
$$f(\alpha)=(a_{n-2},a_{n-3},...,a_0,\alpha)^T,$$

where $\alpha \in R$ be an arbitrary constant. Let us consider a finite square nxn Hankel matrix in the form [11]:

$$H_n=\begin{pmatrix} a_0 & a_1 & \dots & a_{n-1} \\ a_1 & a_2 & \dots & a_n \\ \vdots & \ddots & \ddots & \vdots \\ a_{n-1} & a_n & \dots & a_{2n-2} \end{pmatrix},$$

and let

$$U_n=\begin{pmatrix} 0 & & & \mathbf{O} \\ 1 & 0 & & \\ & . & . & \\ \mathbf{O} & & 1 & 0 \end{pmatrix},\ U_n^T=\begin{pmatrix} 0 & 1 & & \mathbf{O} \\ & 0 & . & \\ & & . & 1 \\ \mathbf{O} & & & 0 \end{pmatrix},$$

$$J_n=\begin{pmatrix} \mathbf{O} & & & 1 \\ & & 1 & \\ & . & & \\ 1 & & & \mathbf{O} \end{pmatrix},$$

$$T_n^C=H_nJ_n \quad \text{and} \quad T_n^R=J_nH_n,$$

where J_n is the matrix of counteridentity [11] and U_n, U_n^T are matrices of the shift operator, respectively. The matrices T_n^C and T_n^R originated from H_n by reordering the columns and rows have the Toeplitz structure.

Let us denote

$$H_nx=e_0, H_ny=e_{n-1}, \tag{1}$$
$$H_ny=e_{n-1}, H_nv=e_{n-2}, \tag{2}$$
$$H_ny=e_{n-1}, H_nw=h(\alpha), \tag{3}$$
$$H_nx=e_0, H_nz=k(\alpha). \tag{4}$$

The vectors x, y, z, w, v are so called *fundamental solutions* of *fundamental equations systems* (1)-(4) corresponding to H_n [11].

Then explicite formulae for a finite Hankel matrix H_n are as follows:

Formula 1. Let the systems (1) be solvable and $y_0 \neq 0$. Then H_n is regular and its inversion can be represented in the form [11]

$$H_n^{-1}=\frac{1}{y_0}\left\{\begin{pmatrix} y_0 & & & \mathbf{O} \\ y_1 & y_0 & & \\ . & & . & \\ y_{n-1} & . & . & y_0 \end{pmatrix}\begin{pmatrix} x_0 & x_1 & . & x_{n-1} \\ x_1 & . & . & \\ . & . & & \\ x_{n-1} & & & \mathbf{O} \end{pmatrix}-\right.$$
$$\left.-\begin{pmatrix} 0 & & & \mathbf{O} \\ x_0 & 0 & & \\ . & & . & \\ x_{n-2} & x_{n-1} & . & 0 \end{pmatrix}\begin{pmatrix} y_1 & y_2 & . & 0 \\ y_2 & . & . & \\ . & . & & \\ 0 & & & \mathbf{O} \end{pmatrix}\right\}.$$

Formula 2. Let the systems (2) be solvable and $y_{n-1} \neq 0$. Then H_n is regular and its inversion can be represented in the form [11]

$$H_n^{-1}=\frac{1}{y_{n-1}}\left\{\begin{pmatrix} y_0y_0 & y_0y_1 & . & y_0y_{n-1} \\ y_1y_0 & y_1y_1 & . & y_1y_{n-1} \\ . & . & . & . \\ y_{n-1}y_0 & y_{n-1}y_1 & . & y_{n-1}y_{n-1} \end{pmatrix}+\right.$$
$$+\begin{pmatrix} v_0 & & & \mathbf{O} \\ v_1 & v_0 & & \\ . & & . & \\ v_{n-1} & . & . & v_0 \end{pmatrix}\begin{pmatrix} y_1 & . & y_{n-1} & 0 \\ . & . & . & \\ y_{n-1} & . & & \\ 0 & & & \mathbf{O} \end{pmatrix}-$$
$$\left.-\begin{pmatrix} y_0 & & & \mathbf{O} \\ y_1 & y_0 & & \\ . & & . & \\ y_{n-1} & . & . & y_0 \end{pmatrix}\begin{pmatrix} v_1 & . & v_{n-1} & 0 \\ . & . & . & \\ v_{n-1} & . & & \\ 0 & & & \mathbf{O} \end{pmatrix}\right\}.$$

Formula 3. Let the systems (3) be solvable. Then H_n is regular and its inversion can be represented in the form [11]

$$H_n^{-1}=\begin{pmatrix} w_0 & & & \mathbf{O} \\ w_1 & w_0 & & \\ . & & . & \\ w_{n-1} & . & . & w_0 \end{pmatrix}\begin{pmatrix} y_1 & y_2 & . & 0 \\ y_2 & . & . & \\ . & . & & \\ 0 & & & \mathbf{O} \end{pmatrix}-$$
$$-\begin{pmatrix} y_0 & & & \mathbf{O} \\ y_1 & y_0 & & \\ . & & . & \\ y_{n-1} & . & . & y_0 \end{pmatrix}\begin{pmatrix} w_1 & w_2 & . & -1 \\ w_2 & . & . & \\ . & . & & \\ -1 & & & \mathbf{O} \end{pmatrix}.$$

Formula 4. Let the equations (4) be solvable. Then H_n is regular and its inversion can be represented in the form [4]

$$H_n^{-1}=\begin{pmatrix} z_{n-1} & . & z_1 & z_0 \\ & . & . & z_1 \\ & & . & . \\ \mathbf{O} & & & z_{n-1} \end{pmatrix}\begin{pmatrix} x_{n-2} & . & x_0 & 0 \\ . & . & . & \\ x_0 & . & & \\ 0 & & & \mathbf{O} \end{pmatrix}-$$

$$- \begin{pmatrix} x_{n-1} & \cdot & x_1 & x_0 \\ & \cdot & \cdot & x_1 \\ & & \cdot & \cdot \\ \mathbf{O} & & & x_{n-1} \end{pmatrix} \begin{pmatrix} \mathbf{O} & & & -1 \\ & & \cdot & z_0 \\ & \cdot & \cdot & \cdot \\ -1 & z_0 & \cdot & z_{n-2} \end{pmatrix}.$$

Remark 1. The vectors $h(\alpha)$ and $k(\alpha)$ in the fundamental systems (3) and (4) consist of the elements of H_n. In this case, the inversion formulae do not require any additional assumptions on H_n. On the other hand, the right sides of systems (1) and (2) are simple and do not depend on elements of H_n. However, in the corresponding inversion formulae it is necessary to add some extra assumptions on H_n.

Remark 2. The Formula 4 follows from the property [11] :

$$H_n^{-1} U_n - U_n^T H_n^{-1} \;=\; z\; x^T - x\; z^T,$$

see also [4].

All the inversion formulae listed above are based on the fundamental solutions of (1)-(4) *with the Hankel matrix* H_n. It has been shown that it is possible to suggest similar formulae for the inversion of H_n by the solution of fundamental systems *with the Toeplitz matrices* T_n^C and T_n^R corresponding to H_n [12]. They can be derived by straightforward application of known theorema on the inversion of finite Toeplitz matrices.

Let us consider the *fundamental equations systems*

$$T_n^C x = e_0, T_n^C y = e_{n-1}, \tag{5}$$

$$T_n^C x = e_0, T_n^C u = e_1, \tag{6}$$

$$T_n^C y = e_{n-1}, T_n^C w = h(\alpha), \tag{7}$$

$$T_n^C x = e_0, T_n^C z = k(\alpha), \tag{8}$$

$$T_n^R x = e_0, T_n^R y = e_{n-1}, \tag{9}$$

$$T_n^R x = e_0, T_n^R u = e_1, \tag{10}$$

$$T_n^R y = e_{n-1}, T_n^R w = f(\alpha), \tag{11}$$

$$T_n^R x = e_0, T_n^R z = g(\alpha), \tag{12}$$

where x, y, z, w, u be the *fundamental solutions* of (5)-(12).

Then explicite formulae for a finite Hankel matrix H_n are as follows [4]:

Formula 5. Let the systems (5) be solvable and $x_0 \neq 0$. Then H_n is regular and its inversion can be represented in the form

$$H_n^{-1} = \frac{1}{x_0} \left\{ \begin{pmatrix} x_{n-1} & \cdot & \cdot & x_0 \\ \cdot & \cdot & \cdot & \\ x_1 & x_0 & & \\ x_0 & & & \mathbf{O} \end{pmatrix} \begin{pmatrix} y_{n-1} & y_{n-2} & \cdot & y_0 \\ & y_{n-1} & \cdot & y_1 \\ & & \cdot & \cdot \\ \mathbf{O} & & & y_{n-1} \end{pmatrix} - \right.$$

$$\left. - \begin{pmatrix} y_{n-2} & \cdot & y_0 & 0 \\ \cdot & \cdot & \cdot & \\ y_0 & \cdot & & \\ 0 & & & \mathbf{O} \end{pmatrix} \begin{pmatrix} 0 & x_{n-1} & \cdot & x_1 \\ & 0 & \cdot & x_2 \\ & & \cdot & \cdot \\ \mathbf{O} & & & 0 \end{pmatrix} \right\}.$$

Formula 6. Let the systems (6) be solvable and $x_{n-1} \neq 0$. Then H_n is regular and its inversion can be represented in the form

$$T_n^{-1} = \frac{1}{x_{n-1}} \left\{ \begin{pmatrix} x_{n-1}x_{n-1} & x_{n-1}x_{n-2} & \cdot & x_{n-1}x_0 \\ \cdot & \cdot & \cdot & \cdot \\ x_1 x_{n-1} & x_1 x_{n-2} & \cdot & x_1 x_0 \\ x_0 x_{n-1} & x_0 x_{n-2} & \cdot & x_0 x_0 \end{pmatrix} + \right.$$

$$+ \begin{pmatrix} u_{n-1} & \cdot & \cdot & u_0 \\ \cdot & \cdot & \cdot & \\ u_1 & u_0 & & \\ u_0 & & & \mathbf{O} \end{pmatrix} \begin{pmatrix} 0 & x_{n-1} & \cdot & x_1 \\ & 0 & \cdot & x_2 \\ & & \cdot & \cdot \\ \mathbf{O} & & & 0 \end{pmatrix} -$$

$$\left. - \begin{pmatrix} x_{n-1} & \cdot & \cdot & x_0 \\ \cdot & \cdot & \cdot & \\ x_1 & x_0 & & \\ x_0 & & & \mathbf{O} \end{pmatrix} \begin{pmatrix} 0 & u_{n-1} & \cdot & u_1 \\ & 0 & \cdot & y_2 \\ & & \cdot & \cdot \\ \mathbf{O} & & & 0 \end{pmatrix} \right\}.$$

Formula 7. Let the systems (7) be solvable. Then H_n is regular and its inversion can be represented in the form

$$H_n^{-1} = \begin{pmatrix} \mathbf{O} & & & w_{n-1} \\ & & \cdot & \cdot \\ & w_{n-1} & \cdot & w_1 \\ w_{n-1} & w_{n-2} & \cdot & w_0 \end{pmatrix} \begin{pmatrix} 0 & & & \mathbf{O} \\ y_0 & \cdot & & \\ \cdot & \cdot & \cdot & \\ y_{n-2} & \cdot & y_0 & 0 \end{pmatrix} -$$

$$- \begin{pmatrix} \mathbf{O} & & & y_{n-1} \\ & & \cdot & \cdot \\ & y_{n-1} & & y_1 \\ y_{n-1} & y_{n-2} & \cdot & y_0 \end{pmatrix} \begin{pmatrix} -1 & & & \mathbf{O} \\ w_0 & \cdot & & \\ \cdot & \cdot & \cdot & \\ w_{n-2} & \cdot & w_0 & -1 \end{pmatrix}.$$

Formula 8. Let the systems (8) be solvable. Then H_n is regular and its inversion can be represented in the form

$$H_n^{-1} = \begin{pmatrix} z_{n-1} & \cdot & \cdot & z_0 \\ \cdot & \cdot & \cdot & \\ z_1 & z_0 & & \\ z_0 & & & \mathbf{O} \end{pmatrix} \begin{pmatrix} 0 & x_{n-1} & \cdot & x_1 \\ & 0 & \cdot & x_2 \\ & & \cdot & \cdot \\ \mathbf{O} & & & 0 \end{pmatrix} -$$

$$- \begin{pmatrix} x_{n-1} & \cdot & \cdot & x_0 \\ \cdot & \cdot & \cdot & \\ x_1 & x_0 & & \\ x_0 & & & \mathbf{O} \end{pmatrix} \begin{pmatrix} -1 & z_{n-1} & \cdot & z_1 \\ & -1 & \cdot & z_2 \\ & & \cdot & \cdot \\ \mathbf{O} & & & -1 \end{pmatrix}.$$

Formula 9. Let the systems (9) be solvable and $x_0 \neq 0$. Then H_n is regular and its inversion can be represented in the form

$$H_n^{-1}=\frac{1}{x_0}\left\{\begin{pmatrix} x_0 & & & \mathbf{O} \\ x_1 & x_0 & & \\ \cdot & \cdot & \cdot & \\ x_{n-1} & \cdot & \cdot & x_0 \end{pmatrix}\begin{pmatrix} y_0 & \cdot & y_{n-2} & y_{n-1} \\ y_1 & \cdot & y_{n-1} & \\ \cdot & \cdot & & \\ y_{n-1} & & & \mathbf{O} \end{pmatrix} - \right.$$

$$\left. - \begin{pmatrix} 0 & & & \mathbf{O} \\ y_0 & 0 & & \\ \cdot & & \cdot & \cdot \\ y_{n-2} & \cdot & y_0 & 0 \end{pmatrix}\begin{pmatrix} x_1 & \cdot & x_{n-1} & 0 \\ x_2 & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ 0 & & & \mathbf{O} \end{pmatrix}\right\}.$$

Formula 10. Let the systems (10) be solvable and $x_{n-1} \neq 0$. Then H_n is regular and its inversion can be represented in the form

$$H_n^{-1} = \frac{1}{x_{n-1}}\left\{\begin{pmatrix} x_0x_0 & x_0x_{n-2} & \cdot & x_0x_{n-1} \\ x_1x_0 & x_1x_{n-2} & \cdot & x_1x_{n-1} \\ \cdot & \cdot & \cdot & \cdot \\ x_{n-1}x_0 & x_{n-1}x_{n-2} & \cdot & x_{n-1}x_{n-1} \end{pmatrix} + \right.$$

$$+ \begin{pmatrix} u_0 & & & \mathbf{O} \\ u_1 & u_0 & & \\ \cdot & \cdot & \cdot & \\ u_{n-1} & \cdot & \cdot & u_0 \end{pmatrix}\begin{pmatrix} x_1 & \cdot & x_{n-1} & 0 \\ x_2 & \cdot & 0 & \\ \cdot & \cdot & & \\ 0 & & & \mathbf{O} \end{pmatrix} -$$

$$\left. - \begin{pmatrix} x_0 & & & \mathbf{O} \\ x_1 & x_0 & & \\ \cdot & \cdot & \cdot & \\ x_{n-1} & \cdot & \cdot & x_0 \end{pmatrix}\begin{pmatrix} u_1 & \cdot & u_{n-1} & \\ u_2 & \cdot & 0 & \\ \cdot & \cdot & & \\ 0 & & & \mathbf{O} \end{pmatrix}\right\}.$$

Formula 11. Let the systems (11) be solvable. Then H_n is regular and its inversion can be represented in the form

$$H_n^{-1} = \begin{pmatrix} w_{n-1} & w_{n-2} & \cdot & w_0 \\ & w_{n-1} & \cdot & w_1 \\ & & \cdot & \cdot \\ \mathbf{O} & & & w_{n-1} \end{pmatrix}\begin{pmatrix} \mathbf{O} & & & 0 \\ & & \cdot & y_0 \\ & \cdot & \cdot & \cdot \\ 0 & y_0 & \cdot & y_{n-2} \end{pmatrix} -$$

$$- \begin{pmatrix} y_{n-1} & y_{n-2} & \cdot & y_0 \\ & y_{n-1} & \cdot & y_1 \\ & & \cdot & \cdot \\ \mathbf{O} & & & y_{n-1} \end{pmatrix}\begin{pmatrix} \mathbf{O} & & & -1 \\ & & \cdot & w_0 \\ & \cdot & \cdot & \cdot \\ -1 & w_0 & \cdot & w_{n-2} \end{pmatrix}.$$

Formula 12. Let the systems (12) be solvable. Then H_n is regular and its inversion can be represented in the form

$$H_n^{-1} = \begin{pmatrix} z_0 & & & \mathbf{O} \\ z_1 & z_0 & & \\ \cdot & \cdot & \cdot & \\ z_{n-1} & \cdot & \cdot & z_0 \end{pmatrix}\begin{pmatrix} x_1 & \cdot & x_{n-1} & 0 \\ x_2 & \cdot & 0 & \\ \cdot & \cdot & & \\ 0 & & & \mathbf{O} \end{pmatrix} -$$

$$- \begin{pmatrix} x_0 & & & \mathbf{O} \\ x_1 & x_0 & & \\ \cdot & \cdot & \cdot & \\ x_{n-1} & \cdot & \cdot & x_0 \end{pmatrix}\begin{pmatrix} z_1 & \cdot & z_{n-1} & -1 \\ z_2 & \cdot & -1 & \\ \cdot & \cdot & & \\ -1 & & & \mathbf{O} \end{pmatrix}.$$

The inversion formulae listed above are based on theorema on the inversion of finite Toeplitz matices. Their proofs can be easily achieved by straightforward application of the row, or column reordering in known explicite formulae for the Toeplitz matrix inversion. It follows from the theorem on the inversion of matrix product that

$$(T_n^C)^{-1} = (H_nJ_n)^{-1} = J_n^{-1}H_n^{-1} = J_nH_n^{-1},$$

and

$$(T_n^R)^{-1} = (J_nH_n)^{-1} = H_n^{-1}J_n^{-1} = H_n^{-1}J_n.$$

The Formulae 5 and 9 follow from the theorem on Toeplitz matrix inversion from [12] (see also [7, 5]). The Formulae 6 and 10 are the corresponding corollaries of the theorem on Toeplitz matrix inversion from [12] (see also [6]). The Formulae 7, 8, 11, and 12 are derived by straightforward application of the theorem on Toeplitz matrix inversion from [11] (see also [17, 8]).

Remark 3. The Formula 5 which was published in [12] in incorrect form without the row reordering required. Moreover, in its proof the reordering of columns must be replaced by the reordering of rows. This mistake was overlooked also in the later translation of [12] (see the References).

Remark 4. An effective implementation of formulae of the same type as the Formulae 1-12, and fast algorithms for the solution of fundamental systems are discussed e.g. in [11] (see also references in [12, 10, 9, 16, 13]).

3 Conclusion

The explicite formulae for Hankel matrix inversion based on the construction of Hankel and Toeplitz fundamental equations systems are discussed. These formulae differ from each other via their form, complexity and different extra assumptions on corresponding matrix. We review the known formulae, derive the new ones, as well as give the correct formulae which have been published in the incorrect form.

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