Ideals in multisemilattices and multilattices

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Abstract: In [5] we have introduced the concepts of multisemilattice and multilattice as a generalization of the concepts of semilattice and lattice, respectively. In this paper, we give another step and we present the concept of ideal in multisemilattice and multilattice (the ideal concept is demanded by the mathematical and computer sciences) that generalizes the corresponding concept in semilattice and lattice.

Keywords: Poset; Non-deterministic operators; Multisemilattice; Multilattice; Ideal.

1 Introduction

Non-deterministic operators – both of fixed and flexible arity– have proven very useful in some applications. When we say non-deterministic operators, we are referring to functions of A^n in 2^A or of A^* in 2^A, respectively, where A^* is the universal language over A, i.e. $A^* = \bigcup_{n\in\mathbb{N}} A^n$.

The notion of poset has proven very relevant in modern mathematics being the lattice theory the best example. Nevertheless, there exist posets that are not lattices as, for example, in *divisibility* theory, *special relativity* theory, the *Jordan-Hölder* theorem ... These posets, although lacking a proper lattice structure, have some of their properties. M. Benado [1] gave a first approach to a generalization of the lattice structure where the supremum and the infimum are replaced by the set of minimal upper bounds and the set of maximal lower bounds, respectively (in other words, deterministic operators are replaced by non-deterministic operators). This new structure is called *multilattice*. M. Benado use this new structure to work with *Dedekind connections*, *Schreier's refinement* theorem and *evaluation* theory.

Nevertheless, neither the algebraic characterization given by M. Benado nor the one given by J. Hansen [3] allow to define the concept of *multisemilattice* (generalization of semilattice structure) In [5], we introduce a new algebraic characterization of multilattice that solves this problem and, moreover, is a more natural generalization of the characterization of lattices. Therefore, the unique algebraic characterization of multisemilattice that exists in the literature, is the given in [5].

Several fields in Computer Science require working with multilattices and multisemilattices. It is enough to consider that the universal language over a set A, whose elements are called chains, with the relation "to be a subchain of" is a multilattice.

In this paper we are looking for the adequate concept of ideal for multisemilattice and multilattice. This concept is relevant for the applications that we have just comment. The paper is structured as follow:

In section 2 we remember the more important results about non-deterministic operators, multisemilattices and multilattices introduced in [5]. In section 3 we analyze the definition of ideal for posets given by Rachunek [7], and we propose a new and more adequate definition of ideal for multisemilattices. In section 4 we develop the theory of ideals in multilattices. Finally, in section 5, we compare our definition with the one proposed by Johnston [4] in the framework of posets.

2 Multisemilattices and multilattices

In this section we resume the concepts that we introduce in [5]. These contributions are the concept of non-deterministic operator together with their properties, and the multisemilattice and multilattice algebraic structures.

2.1 Non-deterministic operators

In a partially ordered set, the sets of multi-supremum and multi-infimum of a subset are not necessarily unitary. So, it is necessary to consider operators that have a set of elements of the domain as image.

Definition 2.1 *Let A be a set. If $F : A^n \to 2^A$ is a total application then we say that F is a* **non-deterministic operator** *(henceforth ndo) of arity n in A.*
If $F : A^* \to 2^A$ is a total application, then we say that

F is a **non-deterministic operator** *of flexible arity in A, where A^* is the universal language defined in A. If F is an ndo with arity $\rho \in \mathbb{N} \cup \{*\}$ in A and $\varnothing \neq B \subseteq A$, we call* **restriction of** F **to** B*, denoted by $F_{/B}$, to the ndo in B given by $F_{/B}(\alpha) = F(\alpha) \cap B$. We say that F is* **full** *if $F(\alpha) \neq \varnothing$ for all $\alpha \in A^\rho$.*

Definition 2.2 *Let F be a ndo in A and $X \subseteq A$. We say that X is F-***closed** *if $F(X^\rho) \subseteq X$, with $\rho \in \mathbb{N} \cup \{*\}$.*

Definition 2.3 *Let F be a ndo of flexible arity in A.*

1. *F is* **commutative** *if for all $n \in \mathbb{N}$ and all $x_1, \dots, x_n \in A$ we have that $F(x_1, \dots, x_n) = F(x_{\sigma 1}, \dots, x_{\sigma n})$ for all permutations of n elements, σ.*
2. *F is* **associative** *if for all $n \in \mathbb{N}$ and all $x_1, \dots, x_n \in A$ we have that $F(x_1, \dots, x_n) = F(F(x_1, \dots, x_{n-1}), x_n) = F(x_1, F(x_2, \dots, x_n))$*
3. *F is* **idempotent** *if $F(x, \overset{n}{\dots}, x) = \{x\}$, for all $x \in A$ and all $n \in \mathbb{N}$.*

Definition 2.4 *Let F and G be two ndos of flexible arity in A. We say that the pair (F, G) has the* **absorption** *property if for all $\omega \in A^*$ we have that:*

- *If $x \in \omega$, then $G(xy) = \{x\}$ for all $y \in F(\omega)$.*
- *If $x \in \omega$, then $F(xy) = \{x\}$ for all $y \in G(\omega)$.*

Definition 2.5 *Let F be ndo with flexible arity in A. We say that F is* **weakly associative** *if for every $\alpha = \alpha_1\alpha_2\alpha_3 \in A^*$ with $\alpha_2 \neq \varepsilon$ and every $z \in A$ it satisfies that: if $F(\alpha_2) = \{z\}$, then*

$$F(\alpha_1 F(\alpha_2)\alpha_3) = \bigcap_{\substack{\alpha=\omega_1\omega_2\omega_3 \\ \omega_2\neq\varepsilon}} F(\omega_1 F(\omega_2)\omega_3)$$

2.2 Multisemilattices

We begin by introducing some previous concepts.

Definition 2.6 *Let $(A, \leq)$ be a poset. If $B \subseteq A$, we denote by $Cot^\uparrow(B)$ the set of upper bounds of B and by $Cot_\downarrow(B)$ the set of lower bounds of B. So, we have two operators $Cot^\uparrow, Cot_\downarrow : 2^A \longrightarrow 2^A$ defined as follows:*

$$Cot^\uparrow(B) = \bigcap_{b\in B} [b); \qquad Cot_\downarrow(B) = \bigcap_{b\in B} (b]$$

Definition 2.7 *Let $(A, \leq)$ be a poset, $a \in A$ and $B \subseteq A$*

- *A* **multi-supremum of** *B is a minimal element of $Cot^\uparrow(B)$. We denote by $Multi\text{-}sup(B)$ the set of multi-supremum of B.*
- *A* **multi-infimum of** *B is a maximal element of $Cot_\downarrow(B)$. We denote by $Multi\text{-}inf(B)$ the set of multi-infimum of B.*

Now we introduce the concept of multisemilattice.

Definition 2.8 *An* **ordered** $\vee$**-multisemilattice** *is a poset, $(A, \leq)$, such that for every nonempty finite subset, $H \subseteq A$ we have that:*

$$Cot^\uparrow(H) = \bigcup \{[z) \mid z \in \ Multi\text{-}sup\ (H)\}$$ [1]

For duality, we obtain the definition of **ordered** $\wedge$**-multisemilattice**.

In order to introduce the algebraic characterization of multisemilattice, we define a new property for ndos.

Definition 2.9 *Let F be an ond with flexible arity in A. We say that F has the property of* **comparability** *if for all $\omega \in A^*$ the two following conditions are satisfied:*
comp$_1$*: if $z \in F(\omega)$, then $F(x, z) = \{z\}$ for all $x \in \omega$.*
comp$_2$*: if $z_1, z_2 \in F(\omega)$ and $F(z_1, z_2) = \{z_1\}$, then $z_1 = z_2$.*

Now we can provide the definition of algebraic multisemilattice.

Definition 2.10 *An* **algebraic multisemilattice***, (A, F), is a set A with an ndo F of flexible arity in A, that satisfies the following properties:*
$(MSR1)$ Commutative law.
$(MSR2)$ Weakly associative law.
$(MSR3)$ Idempotency law.
$(MSR4)$ Comparability law.

Theorem 2.11

i) Let $\mathcal{M} = (A, \leq)$ be an ordered $\vee$-multisemilattice. Then, $(A, F_\vee)$ is an algebraic multisemilattice, denoted by $\mathcal{M}^a$, where $F_\vee(x_1, \dots, x_n) = Multi\text{-}sup\ (\{x_1, \dots, x_n\})$.

ii) Let $\mathcal{M} = (A, F_\vee)$ be an algebraic multisemilattice. The set A with the order relation "$x \leq y$ if and only if $F_\vee(x, y) = \{y\}$" is an ordered $\vee$-multisemilattice denoted by $\mathcal{M}^o_\vee$.

iii) If $\mathcal{M} = (A, \leq)$ is an ordered $\vee$-multisemilattice, then $(\mathcal{M}^a)^o_\vee = \mathcal{M}$.

iv) If $\mathcal{M} = (A, F_\vee)$ is an algebraic multisemilattice, then $(\mathcal{M}^o_\vee)^a = \mathcal{M}$.

The dual result is also true.

[1] Notice that we don't need that $Cot^\uparrow(H) \neq \varnothing$.

From now on, we will use the symbol $F_\odot$ to denote both $F_\vee$ and $F_\wedge$. The following result ensures that the associative property reduces multisemilattices to semilattices.

Theorem 2.12 *Let $(A, F_\odot)$ be a multisemilattice. Then A is a semilattice iff $F_\odot$ is associative and full.*

Moreover, if $(A, F_\odot)$ is a bounded multisemilattice, A is a semilattice iff $F_\odot$ verifies the associative property.

Definition 2.13 *Given a multisemilattice $(A, F_\odot)$ and a subset $\varnothing \neq B \subseteq A$, we say that B is* **submultisemilattice** *of A if the restriction of $F_\odot$ to B, $F_{\odot/B}$, provides the structure of multisemilattice to B.*

2.3 Multilattices

We have now all the neccesary elements to approach the study of ordered structures generalizing the lattice structure.

Definition 2.14 *An* **ordered multilattice** *is a poset $(A, \leq)$, such that for every nonempty finite $H \subseteq A$ we have that:*

$$Cot^\uparrow(H) = \bigcup\{[z) \mid z \in \text{Multi-sup}(H)\}$$
$$Cot_\downarrow(H) = \bigcup\{(z] \mid z \in \text{Multi-inf}(H)\}$$

Definition 2.15 *An* **algebraic multilattice***, $(A, F_\vee, F_\wedge)$, is a set A with two ndos $F_\vee$ and $F_\wedge$ in A, that verify the following axioms:*

$(MR1)$ *Commutative laws.*
$(MR2)$ *Weakly associative laws.*
$(MR3)$ *Idempotency laws.*
$(MR4)$ *Comparability laws.*
$(MR5)$ *Absorption law.*

Theorem 2.16

i) Let $\mathcal{M} = (A, \leq)$ be an ordered multilattice. Then, $(A, F_\vee, F_\wedge)$ is an algebraic multilattice denoted by $\mathcal{M}^a$, being $F_\vee(x_1, \dots x_n) = \text{Multi-sup}\{x_1, \dots x_n\}$ and $F_\wedge(x_1, \dots x_n) = \text{Multi-inf}\{x_1, \dots x_n\}$

ii) Let $\mathcal{M} = (A, F_\vee, F_\wedge)$ be an algebraic multilattice. The set A with the order relation given by "$x \leq y$ if and only if $F_\vee(x, y) = \{y\}$" is an ordered multilattice, denoted by $\mathcal{M}^o$.

iii) Given an ordered multilattice $\mathcal{M} = (A, \leq)$, $(\mathcal{M}^a)^o = \mathcal{M}$.

iv) Given an algebraic multilattice $\mathcal{M} = (A, F_\vee, F_\wedge)$, $(\mathcal{M}^o)^a = \mathcal{M}$.

Proposition 2.17 *Let $(A, F_\vee)$ and $(A, F_\wedge)$ be multisemilattices. Then, $(A, F_\vee, F_\wedge)$ is a multilattice if and only if $(F_\vee, F_\wedge)$ it satisfies the property of absorption*

Now, we show that, similarly we have seen for multisemilattices, associativity reduces the multilattices to lattices.

Theorem 2.18 *Let $(A, F_\vee, F_\wedge)$ be a full multilattice. Then, the following conditions are equivalent:*

1. *$F_\vee$ is associative.*
2. *$F_\wedge$ is associative*
3. *$(A, F_\vee, F_\wedge)$ is a lattice.*

Corollary 2.19 *Let $(A, F_\vee, F_\wedge)$ be a bounded multilattice. A is a lattice if and only if either $F_\vee$ or $F_\wedge$ is associative.*

Definition 2.20 *Let $(A, F_\vee, F_\wedge)$ be a multilattice. We say that $\varnothing \neq B \subseteq A$ is a* **submultilattice** *of A if $(B, F_{\vee/B}, F_{\wedge/B})$ is a multilattice.*

3 Ideals in Multisemilattices

In this section we develop the theory of ideals in multisemilattices. As we will see, it is an appropriate extension of the corresponding concept in semilattices.

Definition 3.1 *Let $(A, \leq)$ be an $\vee$-semilattice and $\varnothing \neq I \subseteq A$. We say that I is an* **ideal** *if I is $\vee$-closed and lower closed.*

This definition admits several generalizations to poset. The widely used in the literature is the following, called s-**ideal** by Rachunek in [7]:

Definition 3.2 *Let $(A, \leq)$ be a poset and $\varnothing \neq B \subseteq A$. B is a s-***ideal** *of A if and only if is a directed[2] set and lower closed. We denote by $\mathcal{I}deals^s(A)$ the set of s-ideales of A.*

A s-ideal, B, is **principal** *if it has maximum. That is, if $B = (b]$ for some $b \in B$.*

The following result characterize the s-ideals in a multisemilattice.

Proposition 3.3 *Let $(A, \leq)$ be a $\vee$-multisemilattice and $\varnothing \neq B \subseteq A$. B is an s-ideal if and only if the following two conditions hold:*

a) If $a, b \in B$, then $F_\vee(a, b) \cap B \neq \varnothing$.

b) B is lower closed.

[2]Given a poset $(A, \leq)$, a subset $\varnothing \neq B \subseteq A$ is **directed** if every nonempty finite subset H of B has an upper bound in B

Proof 1 *To prove that B is lower closed is equivalent to prove that the condition a) is necessary and sufficient to ensure that B is directed. The proof that a) is sufficient is immediate. We prove that is necessary:*

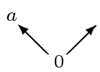
Let B be directed and $a, b \in B$. Then,

$$\begin{aligned} \varnothing \neq Cot^{\uparrow}(\{a,b\}) \cap B &= \left(\bigcup_{x \in F_{\vee}(a,b)} [x)\right) \cap B \\ &= \bigcup_{x \in F_{\vee}(a,b)} ([x) \cap B) \end{aligned}$$

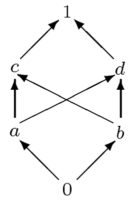
Therefore, there exist $x_0 \in F_{\vee}(a,b)$ with $[x_0) \cap B \neq \varnothing$ and, since B is lower closed, we have that $x_0 \in B$.

The following example shows that the definition of s-ideal for a poset is not adequate in the case of multisemilattices.

EXAMPLE 1.-



$(A, \leq)$



$(B, \leq)$

1. Given the $\vee$-multisemilattice $(A, \leq)$ we have that A is not a s-ideal, because $\{a, b\}$ is not directed.

2. If we consider the multilattice $(B, \leq)$, then $X = \{0, a, b, c\}$ and $Y = \{0, a, b, d\}$ are s-ideals, but $X \cap Y = \{0, a, b\}$ is not a s-ideal.

Therefore, we refuse the option proposed by Rachunek for an arbitrary poset, and we propose the following notion of ideal:

Definition 3.4 *Let $(A, \leq)$ be a $\vee$-multisemilattice and $\varnothing \neq B \subseteq A$. We say that B is an* **ideal** *if B is lower closed and $F_{\vee}$-closed.*

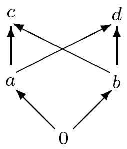
Let $\mathcal{I}deals(A)$ denote the set of all ideals of A and let $\mathcal{I}deals_0(A) = \mathcal{I}deals(A) \cup \{\varnothing\}$. We call $\mathcal{I}deals_0(A)$ the **augmented ideals** *set of A.*

An ideal, B, is **principal** *if it has maximum, that is, if $B = (b]$ for some $b \in B$.*

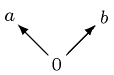
The next example shows that $\mathcal{I}deals(A) \nsubseteq \mathcal{I}deals^s(A)$ and $\mathcal{I}deals^s(A) \nsubseteq \mathcal{I}deals(A)$.

EXAMPLE 2.- Let us considerer the $\vee$-multisemilattices $(A, \leq)$ and $(B, \leq)$ whose diagrams are:

$(A, \leq)$



$(B, \leq)$



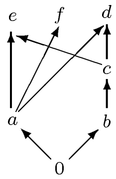
1. $\{0, a, b, c\} \in \mathcal{I}deals^s(A)$; $\{0, a, b, c\} \notin \mathcal{I}deals(A)$

2. $\{0, a, b\} \in \mathcal{I}deals(B)$; $\{0, a, b\} \notin \mathcal{I}deals^s(B)$

The following result is immediate:

Proposition 3.5 *Let A be a $\vee$-multisemilattice. Every ideal, B, of A is a submultisemilattice of A.*

EXAMPLE 3.- Given the $\vee$-multisemilattice $(A, \leq)$ whose diagram is:



The ideals and s-ideals of A are:

- $\mathcal{I}deals(A) = \{(0], (a], (b], (c], (f], \{0, a, b, c, d, e\}, A\}$.

- $\mathcal{I}deals^s(A) = \{(0], (a], (b], (c], (d], (e], (f]\}$.

4 Ideals in Multilattices

In this section, with the same arguments that in the case of multisemilattices, we give a new definition of ideals for multilattices as a generalization of the same concept for lattices.

Definition 4.1 *Let $(A, \leq)$ be an lattice and $\varnothing \neq I \subseteq A$. We say that I is an* **ideal** *if I is an ideal of the $\vee$-semilattice or, equivalently, if the two following conditions hold:*

1. *If $a, b \in I$ then $a \vee b \in I$.*

2. *If $a \in I$ and $x \in A$ then $a \wedge x \in I$.*

Definition 4.2 *Let $(A, \leq)$ be a multilattice. We say that $\varnothing \neq B \subseteq A$ is an* **ideal** *if it is an ideal of the $\vee$-multisemilattice $(A, \leq)$*

The following characterization of ideals is immediate from definition:

Proposition 4.3 *Let $(A, \leq)$ be a multilattice and $\varnothing \neq B \subseteq A$. Then, B is an ideal if and only if the following two conditions hold:*

a) If $a, b \in B$, then $F_{\vee}(a, b) \subseteq B$.

b) If $x \in B$ and $a \in A$, then $F_{\wedge}(a, x) \subseteq B$.

In lattices, the concepts of ideal and s-ideal coincide, that is, if $(A, \leq)$ is a lattice we have that:

$$\mathcal{I}deals(A) = \mathcal{I}deals^s(A)$$

This is not true in multilattices. In example 2 we have that both $(A, \leq)$ and $(B, \leq)$ are multilattices, and $\mathcal{I}deals^s(A) \nsubseteq \mathcal{I}deals(A)$ and $\mathcal{I}deals(B) \nsubseteq \mathcal{I}deals^s(B)$

Notice that our concept of ideal is the natural translation of the characterization of ideals in lattices, and also coincides with the ideal concept in rings.

The following result is immediate:

Proposition 4.4 *Let $(A, \leq)$ be a multilattice. Every ideal is a submultilattice.*

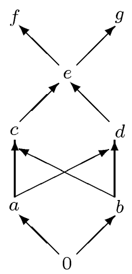
As direct consequence of the definition of ideal, we can ensure:

Proposition 4.5 *Let $(A, \leq)$ be a multilattice. For any two ideals B_1, B_2 in A, we have that $B_1 \cap B_2$ is an ideal of A.*

This allows us to give the following definition:

Definition 4.6 *Let $(A, \leq)$ be a multilattice and $\varnothing \neq X \subseteq A$. We call* **ideal generated by** *X, and we denote by $\mathcal{I}(X)$, to the intersection of all the ideals that contain it.*

Example 4.- Let us considerer the multilattice, $(A, \leq)$, whose diagram is:



$\mathcal{I}(\{c\}) = (e] = \{0, a, b, c, d, e\}$
$\mathcal{I}(\{a, b\}) = (e] = \{0, a, b, c, d, e\}$
$\mathcal{I}(\{f, g\}) = A$

We conclude the section with the result whose realization has guided all our study.

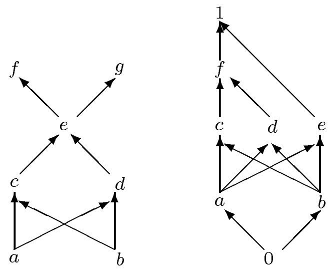
Theorem 4.7 *Let $(A, \leq)$ be a multilattice. Then:*

1. *$(\mathcal{I}deals(A), \subseteq)$ is an associative multilattice.*
2. *$(\mathcal{I}deals_0(A), \subseteq)$ is a lattice.*
3. *If $(A, \leq)$ is bounded, then $(\mathcal{I}deals(A), \subseteq)$ is a lattice.*

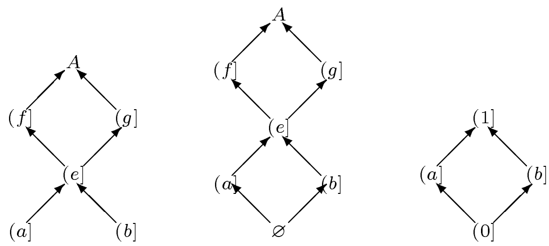
Proof 2 *Let $B_1, B_2 \in \mathcal{I}deals(A)$. If $B_1 \cap B_2 \neq \varnothing$, then $B_1 \cap B_2$ is an ideal and is the infimum of $\{B_1, B_2\}$. On the other hand, since A is an ideal that contains B_1 and B_2, then exists $\mathcal{I}(B_1 \cup B_2)$ and is the supremum of $\{B_1, B_2\}$, and therefore $(\mathcal{I}deals(A), \subseteq)$ is a multilattice. On the other hand, adding the empty set we have the lattice $(\mathcal{I}deals_0(A), \subseteq)$ and therefore we have that $(\mathcal{I}deals(A), \subseteq)$ is associative.*

If $(A, \leq)$ is a bounded multilattice, then every ideal contains $[0]$ that is an ideal and, therefore, $(\mathcal{I}deals(A), \subseteq)$ is a lattice.

Example 5.- Let us consider the multilattices $(A, \leq)$ and $(B, \leq)$ whose diagrams are:



The diagrams of the associative multilattice $(\mathcal{I}deals(A), \subseteq)$ and the lattices $(\mathcal{I}deals_0(A), \subseteq)$ and $(\mathcal{I}deals(B), \subseteq)$ are:



$(\mathcal{I}deals(A), \subseteq)$ $(\mathcal{I}deals_0(A), \subseteq)$ $(\mathcal{I}deals(B), \subseteq)$

5 Other considerations

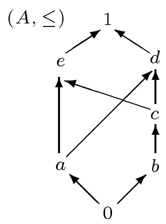
Johnston introduces another concept of ideal in the framework of the posets which is too weak when applied to multisemilattices and multilattices. In this section we analyze this concept in relation with the one we have introduced.

Definition 5.1 [Definition of Johnston [4]] *Let $(A, \leq)$ be a poset and $\varnothing \neq B \subseteq A$. B is a m-***ideal** *if and only if for every nonempty finite subset H of B we have that, if exists $SupH$, then $(SupH] \subseteq B$.*

We denote by $\mathcal{I}deals^m(A)$ the set of m-ideales of a poset A. It is easy to prove that $\mathcal{I}deals^s(A) \subseteq$

$\mathcal{I}deals^m(A)$. Also, as it is shown in the following example, this inclusion is in general strict:

Example 6.- Let $(A, \leq)$ be a poset whose diagram is:



Then, $\{0, a, b\} \in \mathcal{I}deals^m(A)$, but $\{0, a, b\} \notin \mathcal{I}deals^s(A)$

The concept of m-ideal proposed by Johnston, can be characterized in the multisemilattices in the following way:

Proposition 5.2 *Let $(A, \leq)$ be a $\vee$-multisemilattice and $\varnothing \neq B \subseteq A$. Then B is a m-ideal if and only if the following two conditions hold:*

a) If $a, b \in B$ and $F_\vee(a, b) = \{c\}$, then $c \in B$

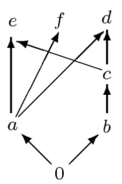
b) B is lower closed.

The following proposition is immediate and ensures that our definition of ideal is stronger that the definition of m-ideal.

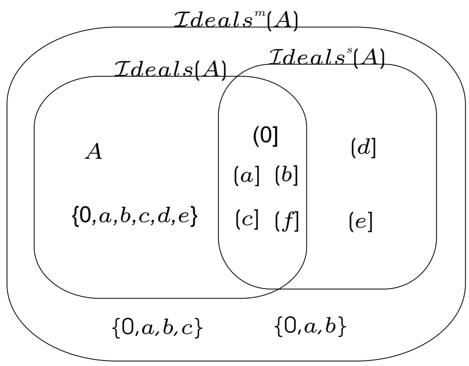
Proposition 5.3 *Let $(A, \leq)$ be a $\vee$-multisemilattice. Then,*

$$\mathcal{I}deals(A) \subseteq \mathcal{I}deals^m(A)$$

Example 7.- Let $(A, \leq)$ be a $\vee$-multisemilattice whose diagram is:

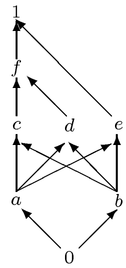


The ideals, s-ideals and m-ideals of A are represented in the following diagram:



From definition 4.6 we have that, if there exists the smallest m-ideal containing a subset $X \neq \varnothing$ of a multilattice, A, denoted by $\mathcal{I}_m(X)$, then $\mathcal{I}_m(X) \subseteq \mathcal{I}(X)$. However, the other inclusion is not true, as we can see in the following example.

Example 8.- Let us consider the multilattice, $(A, \leq)$, whose diagram is:



Then $\mathcal{I}_m(\{c, d\}) = (f] \subsetneq \mathcal{I}(\{c, d\}) = A$.

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