Leslie Discrete–Time Linear Control System and their controlled evolution (*)

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Abstract: In this paper we consider the Leslie matrix model, which describes the time evolution of populations in which fertility and survival rates of individuals strongly depend on their age. This population model is a positive dynamic linear system of age-structured population, in which the state variables represent the number of individuals of the population.
The original Leslie model can be appropriately modified by specifying the input controls, thus obtaining a positive discrete-time linear control system

$$x(k+1) = Lx(k) + Bu(k) \quad k = 0, 1, 2, \ldots$$

where $L \in \mathbb{R}_+^{n\times n}$ is the Leslie matrix, $B \in \mathbb{R}_+^{n\times m}$, and $u(k) \in \mathbb{R}_+^m$ represents the exogenous contribution of immigration or stocking on each age class.
The properties of reachability, controllability and essential reachability of the Leslie system are studied in this work. These properties are characterized in terms of the directed graph associated with the Leslie matrix.

Key-Words: Positive discrete-time linear control system, age-structured populations.

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1 Introduction

Population models are positive dynamic systems in which the state variables represent the number of individuals of the population. Many population models are nonlinear, but an exception is the *Leslie model*, an *age–structured population* where the fertility and the survival rates of individuals strongly depend on their age (or generation). In the Leslie model, the time-period (or year) is denoted by the discrete time variable k, and the state variables $x_1(k)$, $x_2(k)$, ..., $x_n(k)$ denote the number of females (or individuals) of age $1, 2, \ldots, n$, at the beginning of period k. We suppose that there are no differences in the survival rates of males and females, then the process can be expressed by the equations

$$x_{i+1}(k+1) = s_i x_i(k) \quad 1 \le i \le n-1$$

where $0 < s_i \le 1$ is the survival coefficient at age i, that is, the fraction of females of age i that survive at least for one year. The following state equations introduce the *reproduction* process

$$x_1(k+1) = f_1x_1(k) + f_2x_2(k) + \cdots + f_nx_n(k)$$

where $f_i \ge 0$ is the fertility rate of females of age i, that is, the expected number of females born from each female of age i. These equations were proposed by Leslie given a positive linear autonomous model

$$x(k+1) = Lx(k)$$

where $L \geq 0$, is called the *Leslie matrix*

$$L = \begin{pmatrix} f_1 & f_2 & \cdots & f_{n-1} & f_n \\ s_1 & 0 & \cdots & 0 & 0 \\ 0 & s_2 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & s_{n-1} & 0 \end{pmatrix}$$

Leslie models are used for demographic projections, i.e., $x(k) = L^k x(0)$, given $x(0)$. In the case of human populations, by medical progress and socioeconomic conditions, projections cannot be made for long periods of time and the parameters s_i and f_j must be frequently updated.

The initial Leslie model can be modified if we consider the input controls, then a positive discrete-time linear control system is obtained

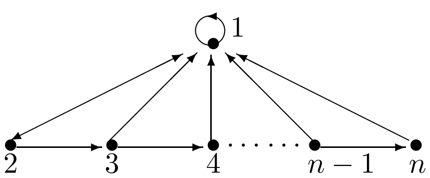
$$x(k+1) = Lx(k) + Gu(k) \quad k = 0, 1, 2, \ldots \quad (1)$$

where $L \in \mathbb{R}_+^{n\times n}$ is the Leslie matrix, and each component of the vector $Gu(k)$ represents the exogenous contribution of immigration on the age class i, that is, we assume that there exists controlled migration from other communities to each generation where $u_i(k) \geq 0$, $1 \leq i \leq n$, $k \geq 0$. This well-structured mathematical model arises in many fields as the dynamic population models with immigration.

The properties of reachability, controllability and essential reachability of the Leslie system are studied in this work. These properties are characterized in terms of the influence graph $D(L)$ associated with the Leslie model.

Definition 1 *The* influence graph $D(L)$ *of the Leslie model is the digraph constructed as follows. The set of vertices of $D(L)$ is denoted by $V = \{1, 2, \ldots, n\}$ and there is an arc (i, j) in $D(L)$ if and only if the entry $l_{ji} > 0$.*

If we consider the survival rates $s_i > 0$, $1 \leq i \leq n-1$, and the fertility rates $f_j > 0$, $1 \leq j \leq n$, therefore $D(L)$ is given by



From the structure of $D(L)$ one can easily prove that the following properties hold. The Leslie matrix is

1. *Irreducible* if and only if the last age class is fertile ($f_n > 0$).

2. *Primitive* if and only if $\gcd\{n - n_j, n_j - n_{j-1}, \ldots, n_2 - n_1\} = 1$, where f_{n_1}, f_{n_2}, ..., f_{n_j}, $n_j \neq n$, are the nonzero fertilities listed in order in the first row of L. In particular, L is primitive if $f_n > 0$ and there are two consecutive fertile age f_{n_1} and f_{n_1+1}.

2 Preliminaries

Consider the open-loop positive discrete-time linear system with vector control sequence

$$x(k+1) = Fx(k) + Gu(k) \quad k = 0, 1, 2, \ldots \quad (2)$$

where $F \in \mathbb{R}_+^{n\times n}$, $G \in \mathbb{R}_+^{n\times m}$, $u(k) \in \mathbb{R}_+^m$ is the control or input vector, and $x(k) \in \mathbb{R}_+^n$ is the state of the system at time k. We denote that positive control system by $(F, G) \geq 0$.

An *i-monomial (column) vector* v is a positive multiple of the unit vector e_i, that is, $v = \alpha e_i$ with $\alpha > 0$. An $n \times m$ real matrix M is an *m-monomial matrix* if it consists of linearly independent monomial columns, i.e., $M = DP$ where D is a nonsingular diagonal matrix and P is a permutation matrix.

We study the structural properties of (positive) reachability, null-controllability, controllability and essential reachability.

Definition 2 *The system in (2) represented by $(F, G) \geq 0$ is said to be (see [2, 6])*

1. reachable *(or controllable from the origin) if for any state $x_f \in \mathbb{R}_+^n$, $x_f \neq 0$, there exist some finite k and a nonnegative input sequence $u(t) \geq 0$, $t = 0, 1, \ldots, k-1$ transferring the state of the system from the origin at $t = 0$ to x_f at time k.*

2. null-controllable *(or controllable to the origin) if for any state $x_f \in \mathbb{R}_+^n$, $x_f \neq 0$, there exist some finite k and a nonnegative input sequence $u(t) \geq 0$, $t = 0, 1, \ldots, k-1$ transferring the*

state of the system from x_f to the origin at time k.

3. (completely) controllable *if for any pair of nonnegative states x_0 and x_f, there exist some finite k and a nonnegative input sequence $u(t) \geq 0$, $t = 0, 1, \ldots, k-1$ transferring the state of the system from x_0 at time 0 to x_f at time k.*

To study these structural properties, we introduce the *reachability matrix* in k-steps defined as $\Re_k(F, G) = \left[G \mid FG \mid F^2G \mid \cdots \mid F^{k-1}G\right]$ and the reachability cone in k-steps defined as the polyhedral cone generated by the columns of $\Re_k(F, G)$, which is denoted by $R_k(F, G)$. This cone is the set of all nonnegative states $x \in \mathbb{R}^n_+$ which are reachable in k-steps, by means of a suitable sequence of nonnegative inputs $\{u(s), s = 0, 1, \ldots, t-1\}$. Then

$$R_\infty(F, G) = \bigcup_{n=1}^{\infty} R_k(F, G)$$

is the set of all the reachable states in finite time. It was obtained [2] that the pair $(F, G) \geq 0$ is reachable if and only if $R_\infty(F, G) = \mathbb{R}^n_+$, or equivalently, the reachability matrix $\Re_k(F, G)$ has an $n \times n$ monomial submatrix.

When not all nonnegative states can be reached in finite time, one considers the essential reachability property. A positive control system $(F, G) \geq 0$ is said to be *essentially reachable* if all positive states are asymptotically reachable (see [2]). From the construction of the reachability cones, one can say that $(F, G) \geq 0$ is essentially reachable if and only if $\overline{R_\infty(F, G)} = \mathbb{R}^n_+$, that is, the states not reachable in a finite number of steps are limits of states which are reachable in a finite number of steps.

Let $D(F)$ be the influence graph of an $n \times n$ nonnegative matrix F constructed as Definition 1. Valcher [7] has provided reachability, controllability and essential reachability criteria by using the notion of *deterministic path* in $D(F)$, that is, a path (set of arcs) in which from each vertex there is at most one outgoing arc, except possibly for the last vertex. It is known that if the pair $(F, G) \geq 0$ is reachable, then the matrix $[F \mid G]$ has an $n \times n$ monomial submatrix. Recently, Bru, Romero and Sánchez [1] establish a converse result, i.e., they add a condition to the existence of the monomial submatrix in order to obtain a characterization of the property of reachability (Proposition 2).

The main idea of [1, Section 2] is to partition the set of vertices $V = \{1, 2, \ldots, n\}$ of $D(F)$ in five different kinds of deterministic paths that they denoted by the sets A, B, C, D_1 and D_2. If we consider the pair $(F, G) \geq 0$, then A denotes the set of all vertices in some different deterministic paths starting from vertices associated with monomial column vectors of G. Next, they consider two different kinds of deterministic paths starting from vertices which are not associated with monomial vectors of G, and construct the sets B and C.

Proposition 1 *[1, Proposition 1] Consider the positive linear system in (2) such that $[F \mid G]$ has an $n \times n$ monomial submatrix. For every $i \in B$ ($i \in C$) there exists a unique deterministic path containing i. Moreover, the deterministic path is a circuit (closed path) and all its vertices are in B (C).*

Proposition 2 *[1, Theorem 1] The positive linear system in (2) is reachable if and only if the matrix $[F \mid G]$ has an $n \times n$ monomial submatrix and the set $A \cup B \cup C = \{1, 2, \ldots, n\}$.*

Proposition 3 *[5, Proposition 2] The following are equivalent:*

1. *The positive linear system in (2) is null-controllable*
2. *F is a nilpotent matrix*
3. *There is no cycles in the digraph $D(F)$*

Proposition 4 *[2, 6] The positive linear system in (2) is controllable if and only if it is reachable and null-controllable.*

Note that the essential reachability is studied in [1] by using two different kinds of circuits denoted by D_1 and D_2 whose vertices are not in $A \cup B \cup C$ and whose communicating classes (blocks in F) have a spectral radius with bordering properties.

Proposition 5 *[1, Theorem 8] The positive linear system in (2) is essentially reachable if and only if the matrix $[F \mid G]$ has an $n \times n$ monomial submatrix and the set $A \cup B \cup C \cup D_1 \cup D_2 = \{1, 2, \dots, n\}$.*

Note that a reachable positive linear system is also essentially reachable.

3 Main Results

Consider the Leslie discrete-time linear control system given in (1). We apply the propositions introduced in Section 2 in order to control the evolution of the Leslie dynamic model of population. Since the entries f_i, $1 \le i \le n$, in the Leslie matrix can be zero, we study different cases depending on these coefficients. From now on, we denote by M an $n \times n$ monomial submatrix of the matrix $[L \mid G]$.

3.1 Case 1

Suppose that $L = M$. This particular case is possible when the entries

$$f_1 = f_2 = \ldots = f_{n-1} = 0, \quad \text{and} \quad f_n > 0,$$

that is, the first $n-1$ generations (or ages) need some time to mature and so do not contribute to the development of new-generations, and only the age n has positive fertile rate. Therefore,

$$x_1(k+1) = f_n x_n(k)$$

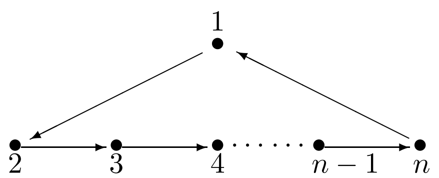
and

$$x_{i+1}(k+1) = s_i x_i(k) \quad 1 \le i \le n-1$$

The Leslie matrix is given by

$$L = \begin{pmatrix} 0 & 0 & \cdots & 0 & f_n \\ s_1 & 0 & \cdots & 0 & 0 \\ 0 & s_2 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & s_{n-1} & 0 \end{pmatrix}$$

and its influence graph $D(L)$ is a cycle of length n



By Proposition 3 the Leslie system is not null-controllable and by Proposition 4 it is not controllable. With respect to the reachability and essential reachability properties we have two possibilities:

1. Any column vector of the control matrix G is not a monomial column vector. In this case, we obtain that the set of vertices in [1, Section 2] are empty, that is, $A = B = C = D_1 = D_2 = \emptyset$, then by Propositions 2 and 5 the Leslie system is neither reachable nor essentially reachable.

2. $G \supset [e_j]$, that is, the control matrix G contains, at least, a j-monomial vector. Then, the set $A = \{1, 2, \dots, n\}$ because we can construct the deterministic path from vertex j to vertex $j-1$ in $D(L)$. By Proposition 2 the Leslie system is reachable.

3.2 Case 2

Consider the general case when $L \neq M$ and suppose that

$$[L \mid G] \supset M \tag{3}$$

therefore G has, at least, a monomial column vector. We study two possibilities:

1. $f_n > 0$, i.e., L has no any zero column (L is irreducible). Similar to Case 1 the influence graph $D(L)$ has a cycle of length n, then the Leslie system is neither null-controllable nor controllable.

   Different from Case 1, consider that $f_{i_1} > 0$, $f_{i_2} > 0$, ..., $f_{i_r} > 0$, $1 \le i_s \le n-1$. By (3) we observe that $G \supset [e_{i_1+1}, e_{i_2+1}, \dots, e_{i_r+1}]$ and we construct these different deterministic paths in $D(L)$

$$\begin{array}{l} \alpha_1^1 = i_1 + 1 \to \alpha_1^2 = i_1 + 2 \to \cdots \to i_2 \\ \alpha_2^1 = i_2 + 1 \to \alpha_2^2 = i_2 + 2 \to \cdots \to i_3 \\ \quad \cdots \quad \cdots \quad \cdots \\ \alpha_r^1 = i_r + 1 \to \alpha_r^2 = i_r + 2 \to \cdots \\ \quad \cdots \to n \to 1 \to \cdots \to i_1 \end{array}$$

   then $A = \{1, 2, \dots, n\}$ and by Proposition 2 the Leslie system is reachable.

2. $f_n = 0$, i.e., L has the last zero column. By (3) we have $G \supset [e_1]$. Furthermore, if the entry $f_i > 0$, for some $1 \leq i \leq n-1$, we have $G \supset [e_1, e_{i+1}]$. Let us consider different cases:

   (a) $f_i = 0$, for every $1 \leq i \leq n-1$, that is, $G \supset [e_1]$ and $D(L)$ is given by a deterministic path. One can obtain that $A = \{1, 2, \ldots, n\}$ and by Propositions 2, 3, 4 the Leslie system is controllable. This case represents the totally sterilized population and it will become extinct in n periods if there is no migration flow.

   (b) If $f_{i_1} > 0$, $f_{i_2} > 0$, ..., $f_{i_r} > 0$, $1 \leq i_s \leq n-1$, then

   $$G \supset [e_1, e_{i_1+1}, e_{i_2+1}, \ldots, e_{i_r+1}]$$

   By using the existence of $r+1$ deterministic paths in $D(L)$ we have $A = \{1, 2, \ldots, n\}$ and by Proposition 2 the Leslie system is reachable. Since $D(L)$ has, at least, a cycle then the Leslie system is neither null-controllable nor controllable (Propositions 3 and 4).

We point out that James and Rumchev [5] studied, in a different way, the properties of reachability, null-controllability, and controllability of the Leslie system, but they only consider the particular case when $G = I$, the $n \times n$ identity matrix. Note that with our discussion we can obtain the following results

Theorem 1 *Consider the Leslie discrete-time linear control system given in (1) and the sets of deterministic paths and cycles in $D(L)$ given in [1, Section 2], then for the Leslie system we always have that the following sets satisfy*

$$B = C = D_1 = D_2 = \emptyset.$$

Theorem 2 *Consider the Leslie discrete-time linear control system given in (1). The Leslie system is reachable if and only if the Leslie system is essentially reachable.*

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