An exact solution of a 3D spiral flow model for a viscous fluid in a stationary porous pipe with application to blood vessels

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Abstract: - There are several attempts to give exact solutions of the Navier-Stokes equations. Certain progress has been made in that direction, with the exception of the three dimensional case with two independent variables. In this note, a new exact solution of the Navier-Stokes equations in conjunction with the continuity equation is proposed, describing the characteristics of three-dimensional axi-symmetric pipe flows with variable suction and injection at the porous pipe walls, with application to blood flow. To solve these equations, it is assumed that the effect of the body force by mass transfer phenomena is the 'porosity' of the porous pipe in which the fluid moves. The resultant of the forces in the pores can be expressed as filtration resistance. The developed solutions are of general application and can be applied to any swirling flow in porous pipes.

Key Words: - exact solution, Navier-Stokes, blood flow, injection suction flow, permeability

1 Introduction

The effect of porous boundaries on steady laminar flow as well as on species concentration profiles has been considered for several different shapes and systems [1-3]. In certain physical and physiological processes filtration and mass transfer occurs as a fluid flows through a permeable tube[5-6]. The velocity and pressure fields in these situations differ from simple Poiseuille flow in an impermeable tube since the fluid in contact with the wall has a normal velocity component. In most cases, the Navier-Stokes equations are reduced to ordinary non-linear differential equations of third order for which approximate solutions are obtained by a mixture of analytical and numerical methods [6,7,8].

In this study, an exact solution of the Navier-Stokes equations is proposed, describing the flow in a porous pipe allowing the suction or injection at the wall to vary with axial distance. In the current research work, a new exact solution of Terrill's proposed phenomenology [9] is presented similar to the model of blood flow through a porous pipe with variable injection and suction at the walls. In the new flow model, a variation of the solutions with Bessel functions based on Terrill's theoretical flow models is adopted. This study uses biomechanical procedures to find exact solutions of the Navier-Stokes equations, governing steady porous pipe flows of a viscous incompressible fluid in a three-dimensional case with two variables, including body force term.

2 The governing equations

The basic equations that describe the mechanics e.g. of blood flow in cardiovascular circulation vessels are the mass conservation equation, and the equations of motion (Navier-Stokes), in a cylindrical polar coordinate system (r,φ,z) where the axis z lies along the centre of the pipe, r is the radial distance and φ is the azimuthal angle.

Starting from the solutions form suggested by Terrill [9] and taking into account body force phenomena, the following solution is proposed. It is considered that in the porous space of the pipe, mass transfer phenomena appear the body force of that is equivalent to the radial pressure gradient. Moreover, when porous spaces exist, a new term is added to the radial pressure gradient which is involved in the first of the Navier-Stokes equations while the following simplified assumptions are made: a) axial symmetry b) the fluid is homogeneous and behaves as a Newtonian fluid c) the pipe is considered of finite length and before the fluid enters the porous pipe its profile has already been developed d) the permeable membrane is treated as a 'fluid medium'.

The Navier-Stokes equations and the continuity equation for the case of the steady axi-symmetric motion of an incompressible fluid in a porous horizontal pipe are:

$$\rho\left(U_r^*\frac{\partial U_r^*}{\partial r^*}-\frac{U_\varphi^{*2}}{r^*}+U_z^*\frac{\partial U_r^*}{\partial z^*}\right)=f_r\rho-\frac{\partial p^*}{\partial r^*}+\mu\left(\frac{\partial^2 U_r^*}{\partial r^{*2}}+\frac{1}{r^*}\frac{\partial U_r^*}{\partial r^*}-\frac{U_r^*}{r^{*2}}+\frac{\partial^2 U_r^*}{\partial z^{*2}}\right) \quad (1)$$

$$\rho\left(U_r^*\frac{\partial U_\varphi^*}{\partial r^*}+\frac{U_r^*U_\varphi^*}{r^*}+U_z^*\frac{\partial U_\varphi^*}{\partial z^*}\right)=\mu\left(\frac{\partial^2 U_\varphi^*}{\partial r^{*2}}+\frac{1}{r^*}\frac{\partial U_\varphi^*}{\partial r^*}-\frac{U_\varphi^*}{r^{*2}}+\frac{\partial^2 U_\varphi^*}{\partial z^{*2}}\right) \quad (2)$$

$$\rho\left(U_r^*\frac{\partial U_z^*}{\partial r^*}+U_z^*\frac{\partial U_z^*}{\partial z^*}\right)=-\frac{\partial p^*}{\partial z^*}+\mu\left(\frac{\partial^2 U_z^*}{\partial r^{*2}}+\frac{1}{r^*}\frac{\partial U_z^*}{\partial r^*}+\frac{\partial^2 U_z^*}{\partial z^{*2}}\right) \quad (3)$$

$$\frac{\partial U_r^*}{\partial r^*}+\frac{U_r^*}{r^*}+\frac{\partial U_z^*}{\partial z^*}=0 \quad (4)$$

where ρ is the mass density, μ is the dynamic viscosity, ν is the kinematic viscosity, U_z^*, U_r^*, U_φ^* are the velocity components in the directions z*, r* (increasing) and φ respectively, f_r, is the body force per unit mass and p^* is the pressure. The above equations can be transformed to dimensionless form by applying the following:

$z^*=zR$ (5)

$r^*=rR$ (6)

$U_z^*=U_zU$ (7)

$U_r^*=U_rU$ (8)

$U_\varphi^*=U_\varphi U$ (9)

$$p^*=p\cdot\rho\cdot U^2 \quad (10)$$

$$U_r\frac{\partial U_r}{\partial r}-\frac{U_\varphi^2}{r}+U_z\frac{\partial U_r}{\partial z}=-(\xi+1)\frac{\partial p}{\partial r}+\left(\frac{\partial^2 U_r}{\partial r^2}+\frac{1}{r}\frac{\partial U_r}{\partial r}-\frac{U_r}{r^2}+\frac{\partial^2 U_r}{\partial z^2}\right)\frac{1}{Re} \quad (11)$$

$$U_r\frac{\partial U_\varphi}{\partial r}+\frac{U_rU_\varphi}{r}+U_z\frac{\partial U_\varphi}{\partial z}=\left(\frac{\partial^2 U_\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial U_\varphi}{\partial r}-\frac{U_\varphi}{r^2}+\frac{\partial^2 U_\varphi}{\partial z^2}\right)\frac{1}{Re} \quad (12)$$

$$U_r\frac{\partial U_z}{\partial r}+U_z\frac{\partial U_z}{\partial z}=-\frac{\partial p}{\partial z}+\left(\frac{\partial^2 U_z}{\partial r^2}+\frac{1}{r}\frac{\partial U_z}{\partial r}+\frac{\partial^2 U_z}{\partial z^2}\right)\frac{1}{Re} \quad (13)$$

$$\frac{\partial U_r}{\partial r}+\frac{U_r}{r}+\frac{\partial U_z}{\partial z}=0 \quad (14)$$

where:

$$f_r\rho=\left(\frac{F_r}{m}\right)\rho=\left(\frac{\rho\dot{V}_\delta U_r^*}{\rho A_\delta\delta}\right)\rho=\left(\frac{\dot{V}_\delta U_r^*}{A_\delta\delta}\right)\rho=-\left(\frac{\dot{V}_\delta k}{A_\delta\delta\mu}\right)\rho\left(\frac{\partial p^*}{\partial r}\right)=-\xi\left(\frac{\partial p^*}{\partial r}\right) \quad (15)$$

and $Re=\dfrac{UR}{\nu}$

For a 'fluid-tissue' system, according to Darcy's law:

$$U_r^*=-\frac{k}{\mu}\left(\frac{\partial p^*}{\partial r^*}\right) \quad (16)$$

with the dimensionless 'porosity' parameter ξ:

$$\xi=\left(\frac{\dot{V}_\delta k\rho}{A_\delta\delta\mu}\right) \quad (17)$$

A_δ is the membrane area, k is the permeability coefficient, m is the mass flow rate across the membrane, $\dot{V}_\delta$ is the volumetric flow rate through the porous space and δ is the thickness of the interstitium. All these data can be calculated from the literature.

3 The proposed solution

Extending the procedure of Terrill, the axial velocity U_z, the radial velocity U_r, and the tangential velocity U_φ are expressed in terms of two functions:

$$U_z=[J_0(br)]e^{-bz} \quad (18)$$

$$U_r=[J_1(br)]e^{-bz} \quad (19)$$

$$U_\varphi=\xi[J_1(br)]e^{-bz} \quad (20)$$

where J_0 and J_1 are the Bessel functions of the first kind and b is the zero of J_0 ($J_0(b)=0$).

The following boundary conditions are satisfied:

a. The no-slip condition at the tube wall:

$U_z = 0$ at r=1 (21)

b. The suction (b>0) or injection (b<0) condition at the pipe axis:

$U_r = 0$ at r=0 (22)

(the speed of suction or injection is assumed to have a value at the walls)

c. The swirl condition at the pipe axis:

$U_\varphi= 0$ at r=0 (23)

(the particles rotation-tangential velocity is assumed to have a value at the walls)

By replacing the forms (18),(19),(20) in the continuity equation and having the known recurrence relations :

$$J_1'(br)=J_0(br)-J_1(br)/(br),\quad J_0'(br)=-J_1(br) \quad (24)$$

the following equations are derived:

$$\frac{\{[J_1(br)]^2e^{-2bz}\}}{r}=\frac{\partial p}{\partial r} \quad (25)$$

$$b\{[J_1(br)]^2+[J_0(br)]^2\}e^{-2bz}=\frac{\partial p}{\partial z} \quad (26)$$

Integration of equation (26) with respect to z gives:

$$0.5\{[J_1(br)]^2+[J_0(br)]^2\}e^{-2bz}=-p(r,z)+\zeta(r) \quad (27)$$

Differentiating the above equation with respect to r and combining the equations (25) and (27) it is finally found :

$$p(r,z)=-0.5\{[J_1(br)]^2+[J_0(br)]^2\}e^{-2bz} \quad (28)$$

Thus, the required solutions for our model are :

$$U_z = [J_0(br)]e^{-bz} \quad (29)$$

$$U_r = [J_1(br)]e^{-bz} \quad (30)$$

$$U_\varphi = \xi[J_1(br)]e^{-bz} \quad (31)$$

$$p(r,z) = -0.5\{[J_1(br)]^2 + [J_0(br)]^2\}e^{-2bz} \quad (32)$$

4 Conclusions

In this note, a new exact solution of the Navier-Stokes equations is proposed, describing the characteristics of three-dimensional axi-symmetric pipe flows with variable suction and injection at the porous pipe walls, with application to blood flow. In fig. 1 the axial velocity distribution across the pipe has been plotted, concerning both the theory of [3] and the presented concept of the exact solution blood flow model with porous wall. The pressure and the pressure gradient are dependent on the radial coordinate r in the porous tube. Body force mechanisms in biological membranes are included because of their importance for mass transport. The body force mechanisms which represents here the volume flow rate in the porous space is strongly connected with the angular velocity (twist of the internal particles). The developed solutions are of general application and can be applied to any swirling flow in porous pipes.

References:

[1] Bugliarello G., Kapur C., Hsiao G., *The profile viscosity and other characteristics of blood flow in a Non-uniform shear Field*, 4th Intern. Cong. Rheology, New-York, 1965.

[2] Bugliarello G., Velocity Distribution and other characteristics of steady and pulsatile blood flow in Fine Glass Tubes, *Biorheology,* 7, 1970, pp 85-107.

[3] Pal D., Rudraiah N., Devanathan R., A couple stress model of blood flow in the microcirculation, *Bulletin of mathematical Rheology,* 4, 1988, pp 329-344.

[4] Fung Y.C., *Biodynamics –Circulation* New-York , Springer-Verlag, 1984.

[5] McDonald D.A., *Blood Flow in Arteries,* London –Arnold, 1974.

[6] Lu C., On the asymptotic solution of laminar channel flow with large suction,. *Siam J. Math. Anal.* , 28 , 1997, pp 1113-1134.

[7] Majdalani J., The oscillatory channel flow with arbitary wall injection. *Zeitschrift fuer angewandte Mathematik und Physik ZAMP,* 52, 2001, pp 33-61.

[8] Taylor C.L., Banks W.H.H., Zaturska M.B., Drazin P.G., Three –dimensional flow in a porous channel, *Quart. J. Mech. Appl. Math.,* 44, 1991, pp 105-133.

[9] Terrill R.M, Laminar flow in a uniformly porous channel with large injection, *Aeronaut. Q.,* 16, 1965, pp 323-332.

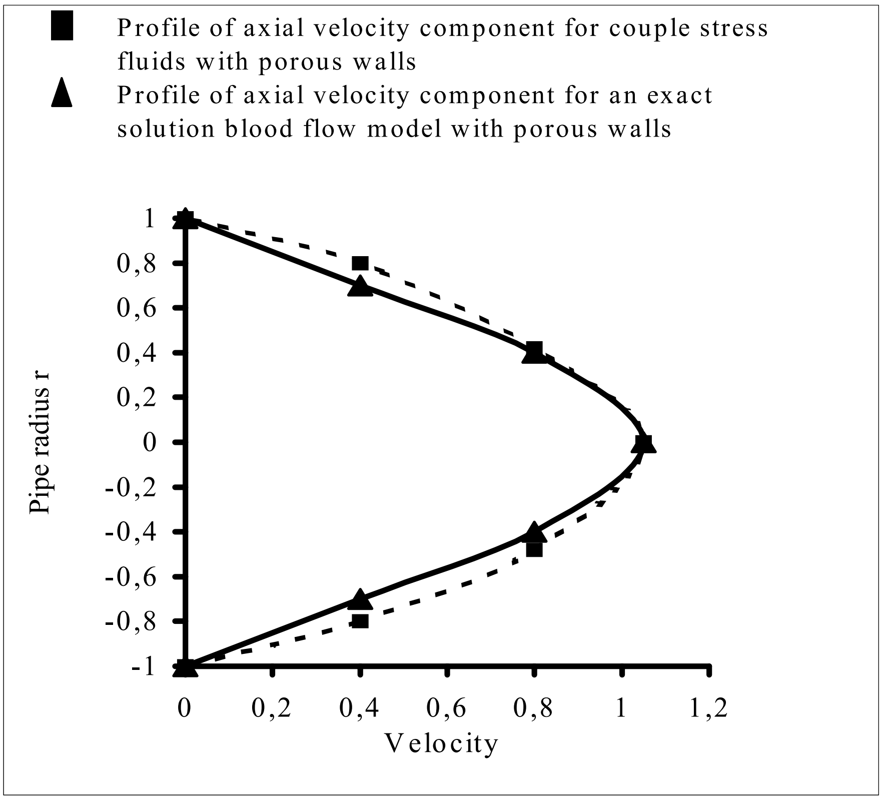

Fig. 1 Velocity di