# Further Investigation of Lyapunov Stability Theory

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*Abstract:* **-** In this paper, the well-known Lyapunov stability theory is further investigated and two new concepts on broad-sense Lyapunov function and Lyapunov distance are introduced. The broad-sense Lyapunov stability theory is then developed. It is shown that a broad-sense Lyapunov function *V(X)* may be positive or negative. If the proposed Lyapunov distance satisfies the specified condition in this paper, the system origin will be asymptotically stable. It is shown that the Lyapunov stability theory is the special case of the proposed broad-sense Lyapunov stability theory. Example is given to explain and verify the concepts on the broad-sense Lyapunov stability theory.

*Key-Words:* Lyapunov stability theory, nonlinear system, control system

## 1 Introduction

Given a control system, the first and most important question about its various properties is whether it is stable, because an unstable control system is typically useless and potentially dangerous. The most useful and general approach for studying the stability of nonlinear control systems is the theory introduced in the late 19$^{th}$ century by the Russian mathematician Alexander Mikhailovich Lyapunov. Lyapunov's work includes two methods for stability analysis: *linearization method* and *direct method* [1]. The linearization method draws conclusions about a nonlinear system's local stability around an equilibrium point from the stability properties of its linear approximation. The direct method determines the stability properties of a nonlinear system by contracting a scalar "energy-like" function for the system and examining the function's time variation. Together, the linearization method and the direct method constitute the so-called Lyapunov stability theory [1]. Since the early 1960's, Lyapunov stability theory has been widely used in control engineering to analyse the stability of control systems and to design controllers for both linear and nonlinear system with the Lyapunov-sense stability.

The objective of this paper is to further investigate the Lyapunov stability theory and two new concepts on broad-sense Lyapunov function and Lyapunov distance are introduced. The broad-sense Lyapunov stability theory is then developed. It is shown that a broad-sense Lyapunov function may be positive or negative, and if the Lyapunov distance satisfies certain condition that will be discussed in the later section, the system origin will be asymptotically stable. It is shown that the Lyapunov stability theory is the special case of the proposed broad-sense Lyapunov stability theory. Example is given to explain the concepts on the broad-sense Lyapunov stability theory.

The organization of this paper is as follows. Section 2 presents a brief review of Lyapunov stability theory. Section 3 further investigates the Lyapunov stability theory. The new concepts of Lyapunov distance and broad-sense Lyapunov function are introduced in Section 4. Example to illustrate the above concepts is presented in Section 5. Section 6 concludes this paper.

## 2. A Brief Review of Lyapunov Stability Theory

Before further investigating the Lyapunov stability theory, let review the simple background issues of the Lyapunov stability theory.

A nonlinear dynamic system can usually be presented by a set of nonlinear differential equations in the form

$$\dot{X} = f(X,t) \qquad (2.1)$$

where X is the nx1 state vector, and $f$ is a nx1 nonlinear vector function

**Definition 2.1:** The nonlinear system (2.1) is said to be autonomous if $f$ does not depend explicitly on time, i.e., if the system's state equation can be written

$$\dot{X} = f(X) \qquad (2.2)$$

where X is the nx1 state vector, and $f$ is a nx1 nonlinear vector function. Otherwise, the system is called non-autonomous.

**Definition 2.2:** A state $X^*$ is an equilibrium state of the system (2.2) if once *X(t)* is equal to $X^*$, it remains equal to $X^*$ for all further time.

Mathematically this means that the constant vector X* satisfies

$$0 = f(X^*) \qquad (2.3)$$

Equilibrium points can be found by solving the nonlinear algebraic equation

**Definition 2.3:** The equilibrium state $X=0$ is said to be stable if for any $R>0$, there exists $r > 0$, such that if $||X(0)|| < r$, then $||X(t)|| < R$ for all $t \geq 0$.

**Definition 2.4**: An equilibrium point $0$ is asymptotically stable if it is stable, and if in addition there exists some $r>0$ such that $||X(0)|| < r$ implies that $X(t) \to 0$ as $t \to \infty$.

**Definition 2.5:** If a function $V(X)$, in a ball centered at the system origin with its radius $R>0$. is positive definite and has continuous partial derivatives and its time derivative along any trajectory of system (2.2) is negative

$$\dot{V}(X) \leq 0 \qquad (2.4)$$

then $V(X)$ is said to be a Lyapunov function for system (2.2).

Based on the above definitions, we can now state two Lyapunov stability theorems:

**Theorem 2.1** (Local stability): If, in a ball $B_{Ro}$, there exists a scalar function $V(X)$ with continuous first partial derivatives such that

*$V(X)$ is positive definite (locally in $B_{Ro}$)*

*$\dot{V}(X)$ is negative semi-definite (locally in $B_{Ro}$)*

*then the equilibrium point 0 is table. If, actually, the derivative $\dot{V}(X)$ is locally negative definite, then the stability is asymptotic.*

**Theorem 2.2** (Global stability): Assume that there exists a scalar function $V(X)$ with continuous first partial derivatives such that

*$V(X)$ is positive definite*

*$V\dot{V} + x\dot{x}$ is negative semi-definite*

*$V(X) \to \infty$ as $||X|| \to \infty$*

Then the equilibrium point at the origin is globally asymptotically stable.

## 3. Further Analysis of Lyapunov Stability Theory

Now let us do further analysis for the Lyapunov stability theory. First, according to the theorem 2.2, if $V(X)$ is a Lyapunov function ($V > 0$ and $\dot{V}(X) < 0$), the equilibrium point 0 is a asymptotically stable. This means that, when the time tends to infinity, the distance $Op$ will reduce to zero or $d(Op)/dt < 0$. Therefore, mathematically we have

$$\frac{d(Op)}{dt} = \frac{V\dot{V} + x\dot{x}}{\sqrt{V^2 + x^2}} < 0 \qquad (3.1)$$

where

$$Op = \sqrt{V^2 + x^2} \qquad (3.2)$$

Because the value of the distance $Op$ is non-negative, the expression (3.2) can be reduced as

$$V\dot{V} + x\dot{x} < 0 \qquad (3.3)$$

**Remark 3.1:** According to the definitions 2.2 and 2.3, we can easily prove that expression (3.3) is a sufficient condition for the system origin to be asymptotically stable.

**Remark 3.2:** It is seen from the expression (3.3) that in order to reduce the distance $Op$ to zero, it is not necessary for the function $V$ to be positive and $\dot{V}$ to be negative. However, it is necessary for $V\dot{V} + x\dot{x} < 0$. This point may motivate us to think about the extension of Lyapunov stability theory.

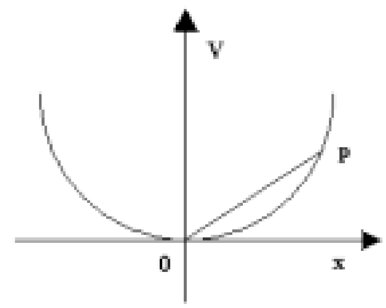


Figure 1: Concept of stability

## 4 Broad-Sense Lyapunov Stability Theory

Now we introduce two new concepts called *Lyapunov distance* and *broad-sense Lyapunov function.*

**Definition 4.1:** If, in a ball $B_{Ro}$, the function V(X) has continuous partial derivatives and its derivative along any state trajectory of system (2.2) satisfies the following inequality

$$V\dot{V} + x\dot{x} < 0 \qquad (4.1)$$

the function V(X) is said to be a broad-sense Lyapunov function and the distance

$$Op = \sqrt{V^2 + x^2} \qquad (4.2)$$

is said to be a Lyapunov distance.

**Remark 4.1:** It is seen that a broad-sense Lyapunov function may not be positive, and a Lyapunov distance has the following property:

$$d(Op)/dt < 0 \tag{4.3}$$

Like an ordinary Lyapunov function, a Lyapunov distance *Op* can be given simple geometrical interpretations. In Figure 1, the distance *Op* is seen to always reduced toward to the system origin, and the state point is seen to move corresponding to the lower and the lower values of the distance *Op*.

## 5 Example

Given a nonlinear system [1]

$$\begin{cases} \dot{x}_1 = x_2 - x_1(x_1^2 + x_2^2) = x_2 - x_1^3 - x_1 x_2^2 \\ \dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2) = -x_1 - x_2 x_1^2 - x_2^3 \end{cases}$$

The origin of the state-space is an equilibrium point for the system. Let $V$ be the negative definite function

$$V\left(X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = -(x_1^2 + x_2^2) < 0$$

The derivative of V along any system trajectory:

$$\dot{V} = -2x_1\dot{x}_1 - 2x_2\dot{x}_2 .$$

We then obtain

$$V\dot{V} + X'\dot{X}$$

$$= V\dot{V} + \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$$

$$= (-x_1^2 - x_2^2)(-2x_1\dot{x}_1 - 2x_2\dot{x}_2) + x_1\dot{x}_1 + x_2\dot{x}_2$$

$$= 2x_1^3\dot{x}_1 + 2x_2^2x_1\dot{x}_1 + 2x_1^2x_2\dot{x}_2 + 2x_2^3\dot{x}_2 + x_1\dot{x}_1 + x_2\dot{x}_2$$

$$= -2x_1^6 - x_1^4 - 6x_1^4x_2^2 - 2x_1^2x_2^2 - 6x_1^2x_2^4 - x_2^4 - 2x_2^6 < 0$$

Therefore, we obtain

$$\frac{dOP}{dt} = \frac{V\dot{V} + X\dot{X}}{\sqrt{V^2 + X^2}} < 0$$

If the initial values of ($x_1$, $x_2$) are (10, -5), the response of the system can be generated and are shown in the figures 2 and 3. This example has shown that a broad-sense Lyapunov function may be negative. Furthermore, if the Lyapunov distance satisfies the condition, *d(OP)/dt < 0*, the system origin will be asymptotically stable. Figures 2 and 3 have verified our investigated.

## 6 Conclusion

This paper has presented a new investigation of well-known Lyapunov stability theory. Two new concepts on broad-sense Lyapunov function and Lyapunov distance have been introduced. The broad-sense Lyapunov stability theory has been developed. It is shown that a broad-sense Lyapunov function *V(X)* may be positive or negative, and if the Lyapunov distance satisfies the condition the condition stated in this paper, the system origin will be asymptotically stable. Authors have showed the Lyapunov stability theory is the special case of the proposed broad-sense Lyapunov stability theory. Example is given to explain the concepts on the broad-sense Lyapunov stability theory. Future work and experiment need to be conducted to further verify the theory and apply it to some applications.

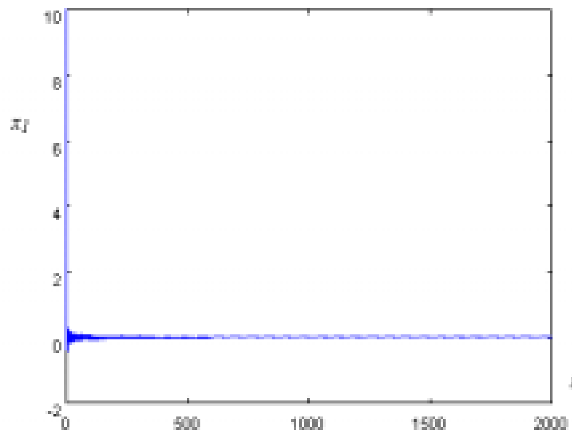


Figure 2: The parameter, $x_1$

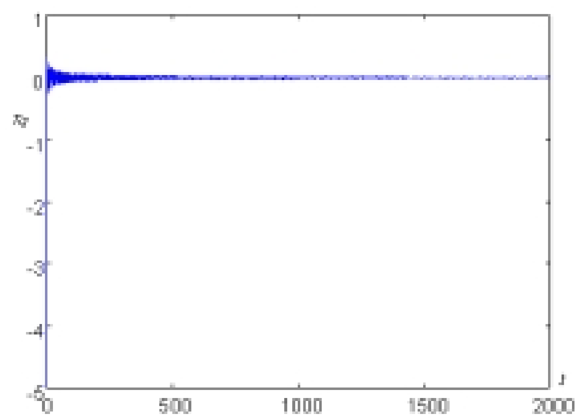


Figure 3: The parameter, $x_2$

*References:*


[1] Slotine, J-J. E. and Li, W. *Applied nonlinear control, Prentice-Hall, Englewood Cliffs, NJ,* 1991.