A Note on the Multiplicative AHP

MIROSLAW KWIESIELEWICZ and EWA van UDEN
Faculty of Electrical and Control Engineering
Technical University of Gdansk
Narutowicza 11/12 Street, 80-952 Gdansk
POLAND

Abstract: The paper concerns the pairwise comparison method used to rank a finite, usually small number of alternatives, in the Analytic Hierarchy Process (AHP), which belongs to a certain class of multiple attribute decision making methods. Ranking of alternatives is calculated based on expert judgments expressing relative importance for each pair of alternatives. From mathematical point of view calculation of weights associated with alternatives leads to approximation of a judgement matrix created based on expert opinions, by a matrix of weights ratios. There are two main approximation techniques: the maximal eigenvalue method and logarithmic least squares one. The study focuses on the former one showing its multiplicative properties. An illustrative numerical example is included.

Key-Words: Pairwise Comparisons, Relative Weights, Logarithmic Regression, Maximal Eigenvalue Method, Multiple Attribute Decision Making, Analytic Hierarchy Process, Rank Preservation

1 Introduction

The decision making is often a process of a choice from a finite number of variants or alternatives and leads to chose the best variant from the point of view of a decision-maker. This choice process should result in achieving a certain goal, which can be specified as a preference order on a variants set. In such case a variant with the highest preference is chosen.

In real life situations we have usually decision problems with multiple criteria, often conflicting with each other, what slightly complicate a process of obtaining a satisfactory solution. For instance a decision problem of buying a car can be considered with respect to a price, traffic conditions, fuel consumption, color, modern conveniences, etc.

Taking into an account the number of variants, the multi-criteria problems can be classified into two groups:

- Multi-goal decision making problems with infinite number of alternatives;
- Multi-attribute decision making problems with usually small, finite number of alternatives.

The paper concerns multi-attribute decision-making.

Coming back to the exemplary car choice problem, it is easy to see that except the criteria (in this case attributes) also sub-criteria can be defined. For instance the traffic conditions can be considered in a city or in a highway. In consequence a hierarchy of attributes can be created and hence the described decision process is called the **A**nalytic **H**ierarchy **P**rocess (AHP) [7].

In the AHP ranking of variants under a given attribute is performed by pairwise comparisons of variants [7]. The comparisons are made by an expert but can be also extended to a group of experts. An expert is asked to provide a relative importance of first variant in a given pair over the second one. Based on his opinions, a judgement matrix is created. It is assumed that the judgment matrix contains relative weights of variants in form of ratios, biased in the elicitation process. Then a calculation of weights leads to an approximation of the matrix of judgement by the matrix of ratios.

Two main techniques are used in the approximation process:

- Maximal eigenvalue method [7];
- Logarithmic least squares method [3,7].

The eigenvalue method often results in a loss of weight [5,6], whilst the logarithmic regression does not cause such phenomenon. The paper recalls a problem of adding a new alternative with the relative weight and shows that the logarithmic least squares method preserves weights. It shows a multiplicative character of weights, what is a very important property.

2 Paired comparisons

Assume that there are n alternatives $F_1, F_2, ..., F_n$ and an expert is asked to provide his opinions concerning each pair of them, expressing intensity of importance of one factor in a pair over the second one with a use of the preference scale from the Table 1. Thus a judgement matrix $\mathbf{R}$ can be created where r_{ij} is an estimate for the relative significance of the factors (F_i, F_j), provided by the expert. In general case a judgement matrix has a form:

$$\mathbf{R}=\begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ r_{n1} & r_{n2} & \cdots & r_{nn} \end{pmatrix}. \tag{1}$$

where:

$$r_{ij} > 0, \quad r_{ij} = 1/r_{ji} \, \forall i, j = 1,2,\ldots,n. \tag{2}$$

Our purpose is to obtain a vector of positive estimates $\mathbf{p}=(p_1,\ldots,p_n)^T$, assuming that the following matrix of ratios:

$$\mathbf{P}=\begin{pmatrix} p_1/p_1 & p_1/p_2 & \cdots & p_1/p_n \\ p_2/p_1 & p_2/p_2 & \cdots & p_2/p_n \\ \vdots & \vdots & \vdots & \vdots \\ p_n/p_1 & p_n/p_2 & \cdots & p_n/p_n \end{pmatrix}. \tag{3}$$

approximates the judgement matrix **R**.

Table 1 Explanation of the scale

Intensity of importance	Definition
1	Indifference
3	Weak preference of one over another
5	Essential or strong preference
7	Very strong or demonstrated preference
9	Absolute preference
2,4,6,8	Intermediate values between adjacent scale values
Reciprocals of above nonzero	If factor *i* has one of the above nonzero numbers assigned to it when compared with factor *j*, then *j* has the reciprocal value when compared with *i*.

There are two main approaches to evaluate ranking, namely the maximal eigenvector [7] and logarithmic least square method [3,7]. The former one has several drawbacks. First of all the solution of the problem is dependent on a judgement matrix transposition [1,2]. Secondly, an adding of a new alternative causes a loss of weight [6]. In other words, when we calculate a ranking based on a judgement matrix, after adding an extra alternative with a certain weight and recalculating ranking, the relative weight is lost.

3 Logarithmic least squares method

The logarithmic least squares method leads to find such a vector **p** which minimizes the distance between matrix **P** and **R** in a sense of Euclidean norm in a logarithmic scale:

$$I=\sum_{i,j>i}^{n}\left(\ln(r_{ij})-\ln\left(\frac{p_i}{p_j}\right)\right)^2. \tag{4}$$

Putting $y_{ij}=\ln(r_{ij})$ and $x_i=\ln(p_i)$ for $\forall i,j=1,\ldots,n$, we get [3]:

$$\min_{x_i,\, i=1,\ldots,n}\left\{I=\sum_{i,j=1}^{n}\left(y_{ij}-x_i+x_j\right)^2\right\}. \tag{5}$$

Assuming that:

$$\prod_{i=1}^{n} p_i = 1 \tag{6}$$

and coming back to exponentials $p_i=\exp(x_i)$, $r_{ij}=\exp(y_{ij})$, $i,j=1,\ldots,n$ we finally get:

$$p_i=\left(\prod_{j=1}^{n} r_{ij}\right)^{\frac{1}{n}}, \quad i=1,\ldots,n. \tag{7}$$

Therefore the logarithmic least squares method is often called the geometric mean one.

4 Properties of the geometric mean method

The geometric mean method in contradistinction to the maximal eigenvalue one gives a unique, geometrically normalized solution independent on the scale inversion [1,2]. Moreover adding a new alternative with a certain weight does not result in the loss of weight [5]. It is easy to show for a given ranking: $\mathbf{p}=(p_1,\ldots,p_n)$, obtained from the matrix **R**, that after adding a new alternative with the weight c in comparison to the first alternative:

$$p_{n+1} = cp_1 \tag{8}$$

and recalculating the ranking based on the matrix:

$$\mathbf{R}'(c) = \begin{pmatrix} r_{11} & r_{12} & \dots & r_{1n} & 1/c \\ \vdots & & & & \vdots \\ r_{n1} & r_{n2} & \dots & r_{nn} & p_n/(cp_1) \\ c & (cp_1)/p_2 & \dots & (cp_1)/p_n & 1 \end{pmatrix}, \tag{9}$$

a new ranking $\mathbf{p}' = (p_1', \dots, p_n', p_{n+1}')$, keeps the relative weight [5]:

$$p_{n+1}' = cp_1' \tag{10}$$

See [5] for more details.

Additionally it can be proved that [4]:

$$p_i' = dp_i, \quad i = 1,2,..,n, \tag{11}$$

where d is a constant. In other words, adding a new alternative gives a new ranking vector proportional to the former one.

Based on the matrix $\mathbf{R}'(c)$ and applying the geometric mean method (7), the new weight vector $\mathbf{p}' = (p_1', \dots, p_n', p_{n+1}')$ can be calculated in the following way:

$$p_1' = \left(\prod_{j=1}^{n} r_{ij} \frac{1}{c} \right)^{\frac{1}{n+1}} = \left(\prod_{j=1}^{n} r_{ij} \frac{p_1}{cp_1} \right)^{\frac{1}{n+1}} \tag{12}$$

and

$$p_i' = \left(\prod_{j=1}^{n} r_{ij} \frac{p_i}{cp_1} \right)^{\frac{1}{n+1}}, \quad i = 2,\dots,n, \tag{13}$$

what in the general case gives:

$$p_i' = \left(\prod_{j=1}^{n} r_{ij} \frac{p_i}{cp_1} \right)^{\frac{1}{n+1}}, \quad i = 1,\dots,n. \tag{14}$$

Applying again the geometric mean method (7) to the equation (14) we get [4]:

$$p_i' = \left(\prod_{j=1}^{n} r_{ij} \frac{p_i}{cp_1} \right)^{\frac{1}{n+1}} = \left(\prod_{j=1}^{n} r_{ij} \frac{\left(\prod_{k}^{n} r_{ik} \right)^{\frac{1}{n}}}{cp_1} \right)^{\frac{1}{n+1}} =$$

$$= \left(\prod_{j=1}^{n} r_{ij} \right)^{\frac{1}{n+1}} \left(\frac{1}{cp_1} \right)^{\frac{1}{n+1}} \left(\prod_{j=1}^{n} r_{ij} \right)^{\frac{1}{n+1} \bullet \frac{1}{n}} =$$

$$= \left(\prod_{j=1}^{n} r_{ij} \right)^{\frac{1}{n+1} + \frac{1}{n(n+1)}} \left(\frac{1}{cp_1} \right)^{\frac{1}{n+1}} = \left(\prod_{j=1}^{n} r_{ij} \right)^{\frac{1}{n}} \left(\frac{1}{cp_1} \right)^{\frac{1}{n+1}} =$$

$$= p_i \left(\frac{1}{cp_1} \right)^{\frac{1}{n+1}},$$

what gives (11) [4]:

$$p_i' = dp_i, \quad i = 1,2,..,n, \tag{15}$$

where:

$$d = \left(\frac{1}{cp_1} \right)^{\frac{1}{n+1}}.$$

5 Numerical example

As an illustration of the approach presented, the example of Monsuur [6] is recalled.

Let us consider a 3x3 judgement matrix

$$\mathbf{R} = \begin{pmatrix} 1 & 5 & 7 \\ \frac{1}{5} & 1 & 6 \\ \frac{1}{7} & \frac{1}{6} & 1 \end{pmatrix}.$$

The priority vector obtained by the geometric mean method has the following form

$$\mathbf{p}(\mathbf{R}) = (3.271, 1.063, 0.288).$$

As described by Monsuur [6], the weight of a fourth alternative may be obtained by comparing it to the weight p_1 of the first alternative. Now, according to (8) and assuming that:

$$p_4 = 0.5 \times p_1 = 1.636\,,$$

for a relative weight c=0.5, the matrix $\mathbf{R}'(0.5)$ will take a form of:

$$\mathbf{R}'(0.5) = \begin{pmatrix} 1 & 5 & 7 & 2 \\ \frac{1}{5} & 1 & 6 & 0.650 \\ \frac{1}{7} & \frac{1}{6} & 1 & 0.176 \\ 0.500 & 1.539 & 5.685 & 1 \end{pmatrix}$$

and the solution vector becomes now

$$\mathbf{p}(\mathbf{R}'(0.5)) = (2.893,\ 0.94,\ 0.254,\ 1.446)$$

As was recalled in the previous section, the relative weight c associated with the new alternative has not been reduced, i.e. the equation (10) is satisfied

$$p'_4 / p'_1 = c = 0.5 \text{ or } p'_4 = 0.5 \times p'_1 = 1.446\,.$$

Denoting

$$\mathbf{p} = (p_1, \ldots, p_n, p_{n+1}) = (3.271{,}1.063{,}0.288{,}1.636)$$

and

$$\mathbf{p}' = (p'_1, \ldots, p'_n, p'_{n+1}) = (2.893{,}0.94{,}0.254{,}1.446)$$

it easy to check that

$$\mathbf{p}' = d\mathbf{p}\,,$$

where

$$d = \left(\frac{1}{cp_1}\right)^{\frac{1}{n+1}} = 0.8844$$

6 Remarks and comments

Ranking of a finite, small number of alternatives with a use of the pairwise comparison method was considered in this paper. To derive weights from paired comparison judgement matrix, a matrix approximation technique based on the logarithmic regression was applied. A situation of adding a new alternative was examined from the point of view of relative weight loss. It was shown that the geometric mean method does not result in a weight loss. A multiplicative property of the solution was shown. Numerical examples were included.

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