On some capabilities of the SVD expansion to handle images

FRANCISCO PEDROCHE SÁNCHEZ
Department de Matemàtica Aplicada
Universitat Politècnica de València
Camí de Vera s/n, 46022 València
Spain

Abstract: - This comunication describes some image-compression concepts such as the transmitted energy of the digital signal and the relative error of the compressed image in function of the number and magnitude of the singular values used in the Singular Value Decomposition of the matrix that represents the original picture. Comparisons are made with wavelet-based techniques using a test image.

Key-Words: - Singular Values, SVD expansion, Matrix rank, Outer product, Image processing, Wavelets.

1 Introduction

One of the most significant developments in scientific computation has been the increased use of the SVD in application areas that require the intelligent handling of matrix rank. Some researchers have studied this topic in connection with the treatment of images in the field of biomedical applications (see [1], [4] and [8] for details). Due to the achievements made in digital signal processing and image coding since 1980 by means of the construction of wavelets to form bases for square-integrable functions, and the great repercussion of that techniques, it is interesting to relate some concepts used in wavelet-based techniques to those used in the SVD-based techniques. The main goal of this comunication is to describe some of these concepts and to compare the results obtained via wavelets and via SVD in a particular test-case.

The structure of the comunication is the following. First, some results and concepts concerning the SVD of a real $m \times n$ matrix are introduced. Next, some image-compression concepts are defined and the application of these concepts in terms of those used in SVD are established. Finally, some results are obtained by applicating these concepts to the treatment of the portable greymap test-image called *Lena* and some conclusions are established.

2 The SVD expansion

Theorem 1. For any real $m \times n$ matrix A, there exist a real factoritzation:

$$A = U\ S\ V^T \tag{1}$$

where U is a $m \times m$ real orthogonal matrix, V is a $n \times n$ real orthogonal matrix , and the $m \times n$ matrix S is real pseudo-diagonal with nonnegative diagonal entries $\sigma_i, i = 1, 2, \ldots, \min\{m, n\}$,called the singular values of the matrix A.

The σ_i are usually sorted in non-increasing order of magnitude. The set of singular values $\{\sigma_i\}$ is called the singular espectrum of the matrix A. The triplet (u_i, σ_i, v_i) is called the i-th singular triplet of A.

Lemma 1. The number of singular values, different from zero, equals the algebraic rank of the matrix A.

Lemma 2. Via the SVD, any real $m \times n$ matrix A can be written as the sum of $r = rank(A)$ rank one matrices:

$$A = \sum_{i=1}^{r} \sigma_i u_i v_i^T \tag{2}$$

where the (u_i, σ_i, v_i) is the i-th singular triplet of the matrix A. The representation (2) is referred to as the *SVD expansion* or the *outer-product expansion* or the *dyadic decomposition* of A. Following [5] it is usually to denote

$$A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T \tag{3}$$

whence $k < r$, the rank-*k* expansion matrix.

Lemma 3. Frobenius norm of $m \times n$ matrix A of rank r:

$$\|A\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2} = \sqrt{\sum_{i=1}^{r} \sigma_i^2} \tag{4}$$

where the σ_i are the singular values of A.Proofs of the above Theorem and Lemmas can be found in references [2], [3], [5], [6] and [7].

2 Connection with image-processing concepts

Suppose there are $(r-k)$ small singular values of A that can be neglected. Then, the matrix A_k can be an adequate approximation of A in digital processing applications.

Definition 1. Consider the sequence of vectors $a_i \in \Re^{m\times 1}$, $i=1,\ldots,n$,and the associated $m\times n$ matrix $A=\begin{pmatrix}a_1 & a_2 & \ldots & a_n\end{pmatrix}$. Its total energy $E[A]$ is defined via the Frobenius norm of A:

$$E[A]=\|A\|_F^2 \tag{5}$$

This definition corresponds to that of the energy of a discrete image stored as an array A (see [9] for details).

Definition 2. The relative amount of energy of A that is retained by A_k is given by the quotient:

$$\frac{E[A_k]}{E[A]}=\frac{\|A_k\|_F^2}{\|A\|_F^2}=\frac{\sum_{i=1}^{k}\sigma_i^2}{\sum_{i=1}^{r}\sigma_i^2} \tag{6}$$

where equations (2), (3) ,(4) and (5) and the fact that $rank(A_k)=k$, have been used to set the last equality.

By compressing a picture it is meant, in image-processing terminology, converting the image stored in A into a new format that requires less bits to transmit. Hence, A_k can be considered as a *rank-k* compressed version of A that captures much of the energy of A as possible.

Definition 3. In order to express how accurately a compressed image stored as a $m\times n$ matrix $C=\{c_{ij}\}$approximates the original image $A=\{a_{ij}\}$, the *relative difference* $D(A,C)$ is defined by:

$$D(A,C)=\frac{\sqrt{\sum_{i,j=1}^{m,n}\left(a_{ij}-c_{ij}\right)^2}}{\sqrt{\sum_{i,j=1}^{m,n}a_{ij}^2}} \tag{7}$$

As a practical rule, if $D(A,C)\le 0.05$ then matrix C is an acceptable approximation to A (see [9]).

Theorem 2. If A_k is taken as a compressed version of A, the relative difference $D(A,A_k)$ is given by:

$$D(A,C)=\frac{\sqrt{\sum_{i=k+1}^{r}\sigma_i^2}}{\sqrt{\sum_{i=1}^{r}\sigma_i^2}} \tag{8}$$

Proof. From Lemma 2 we have:

$$A=A_k+\sum_{i=k+1}^{r}\sigma_i u_i v_i^T \tag{9}$$

and then, the matrix $A-A_k$ admits the following Singular Value Decomposition:

$$\begin{pmatrix}u_{k+1} & \cdots & u_r\end{pmatrix}\begin{pmatrix}\sigma_{k+1} & & \\ & \ddots & \\ & & \sigma_r\end{pmatrix}\begin{pmatrix}v_{k+1}^T\\ \vdots \\ v_r^T\end{pmatrix} \tag{10}$$

and from Lemma 3, we conclude that

$$\|A-A_k\|_F=\sqrt{\sum_{i=k+1}^{r}\sigma_i^2} \tag{11}$$

Now, denoting by $\{c_{ij}\}$ the elements of A and applicating Definition 3, Lemma 3 and equation (11) we have:

$$\begin{aligned} D(A,C)&=\frac{\sqrt{\sum_{i,j=1}^{m,n}\left(a_{ij}-c_{ij}\right)^2}}{\sqrt{\sum_{i,j=1}^{m,n}a_{ij}^2}}= \\ &=\frac{\|A-A_k\|_F}{\|A\|_F}=\frac{\sqrt{\sum_{i=k+1}^{r}\sigma_i^2}}{\sqrt{\sum_{i=1}^{r}\sigma_i^2}} \end{aligned} \tag{12}$$

which completes the proof. Notice that expression (8) gives a mesure of the error in terms of the singular values of A involved in the expansion.

3 Results

We use a standard test image, know as *Lena*, which appears frequently in the field of image processing, stored as a 512 $\times$512 matrix A. This image is shown in figure 1. Its values are grey-scale intensity values from 0 to 255. Compressing this image using the 4-

level Coifflet-12 wavelet transform, a compression ratio of 16:1 can be achieved with a relative difference $D(A,C) = 0.040$ (that ratio can be improved, see [9] for details). In order to achieve $D(A,C) \leq 0.040$ via the SVD it is necessary, according to expression (8), to make an expansion of k terms such as:

$$\sum_{i=k+1}^{r} \sigma_i^2 \leq (0.040)^2 \sum_{i=1}^{r} \sigma_i^2 \qquad (13)$$

Using MATLAB to evaluate the SVD, the singular spectrum of the matrix A can be calculated. We obtain $r = rank(A) = 507$, and $k \geq 80$ to fulfill equation (13). Then, to obtain a relative difference less or equal to 0.040 we may use A_{80} as a compressed version of A. The image obtained from A_{80} is shown in figure 2. Using u_i and $\sigma_i v_i^T, i = 1, \cdots, 80$, to transmit A_{80}, a compression ratio of $3:1$ is achieved. In figures 3, 4 and 5 we show images from A_{10}, A_{15} and A_{70}.

Figure 1: Image from matrix A.

Figure 2: Image from A_{80}.

Figure 3: Image from A_{10}.

Figure 4: Image from A_{15}.

Figure 5: Image from A_{70}.

In figure 6 the plot of the transmitted energy against k obtained from equation (6) is shown. The plot of the relative energy against k derived from equation (8) is also presented, in figure 7, as an example of the applicability of the expressions described above. Notice, for example, that with an approximation of rank one, more than the 90% of the energy of A is transmitted. This figure reaches more than the 99% with a rank $k \geq 50$ approximation.

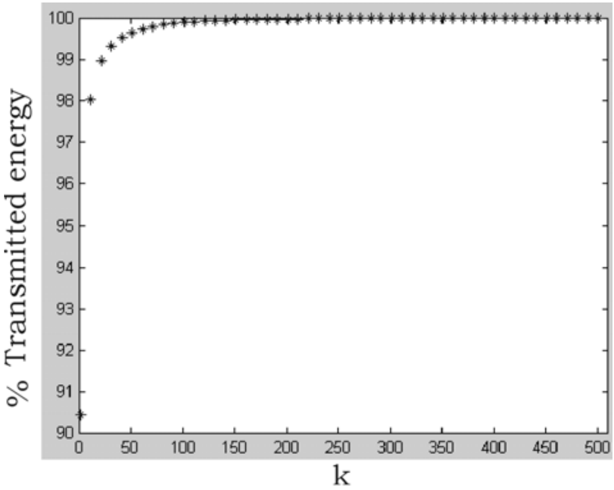

Figure 6: Transmitted energy against *k*.

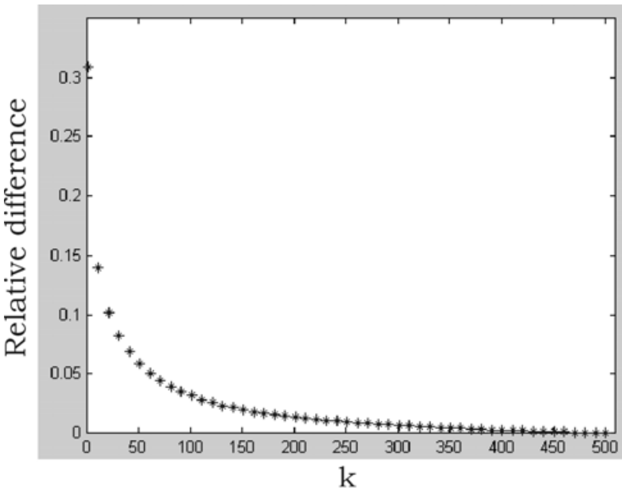

Figure 7: Relative diference against *k*.

4 Conclusions

Definitions of image-treatment theory in terms of matrix-algebra concepts that can be useful for image-SVD applications have been introduced. An expression that gives the relative difference between two images, according to that used on wavelet-based techniques, is described in terms of the Frobenius norm and the number of singular values used in the SVD expansion of A. This expression can be useful in image-treatment problems where the SVD is applicable. In order to optimize the compression ratio, research has to be done in the field of quantization and transmission of the compressed image stored in A_k or in the k vectors of each of the unitari matrices U and V.

Acknowledgements: This work is supported by spanish DGI grant BFM2001-2641.

References:

[1]Callaerts, D. de Moor, B., Vandewalle, J., Sansen, W., Vantrappen, G. and Janssens, J.Comparison of the SVD Methods to extract the foetal electrocardiogram from cutaneous electrode signals. *Med. & Biol. Eng. & Comp., 28,*1990. pp. 217-224.

[2]Datta, B.N. *Numerical Linear Algebra and Applications*. Brooks/Cole Publishing Co. USA. 1995.

[3]Demmel, J.W. *Applied Numerical Linear Algebra*. Society for Industrial and Applied Mathematics. Philadelphia. EEUU. 1997.

[4]De Moor, B., and Vandewalle J. Oriented energy and oriented signal to signal ratios: The (Generalized) Singular Value Decomposition, in *SVD and Signal Processing: Algorithms, Applications and Architectures*. E.F. Deprettere (Ed), North Holland, Elseviers Science Publishers B.V., 1988, pp. 209-232.

[5]Golub, G.H. and Van Loan, C. F., *Matrix Computations*. The Johns Hopkins University Press. 1996.

[6]Lancaster, P. and Tismenetsky, M. *The Theory of Matrices*. Academic Press, New York. 1985.

[7]Trefethen, L.N., and Bau, D. *Numerical Linear Algebra.* Society for Industrial and Applied Mathematics. Philadelphia. EEUU. 1997.

[8]Vidal, V., Ginestar, D., and Verdú, G. Análisis de imágenes mediante descomposición parcial en valores singulares, in *XVII Congreso de Ecuaciones Diferenciales y Aplicaciones-VII Congreso de Matemática Aplicada*. L. Ferragut, y A. Santos (Ed). Departamento de Matemática Aplicada, Universidad de Salamanca.Spain. 2001. p. 475

[9]Walker, J.S. *A primer on wavelets and their scientific applications*. Chapman & Hall/CRC. USA. 1999.

[10]Wickerhauser, M. V. Comparison of picture compression methods: wavelet, wavelet packet and local cosine transform coding, in *Wavelets: Theory, Algorithms, and Applications*, Charles K. Chui, Laura Montefuso and Luigia Puccio (Eds). Academic Press, Inc. EEUU.J.S. 1994. pp. 585-621.