Generalized variational framework for analysis of local projectional methods solving linear equations of particle transport

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Abstract: In the paper the theory of bilinear forms suitable for the equation of particle neutron transport is developed. These bilinear forms are bounded with respect to both their arguments, each of them belonging to an approximate functional space. In general the spaces do not coincide. Moreover these bilinear forms are not coercive in standard sense. The proposed variational formulation is extended so that one may easily handle issues of domain decomposition methods and local discontinuous projectional methods solving the transport equation.

Key-Words: Particle transport equation, variational formulation, local projection methods.

1. Introduction

The original discontinuous Galerkin (DG) finite element method was introduced by Reed and Hill [8] for solving the neutron transport equation.

LeSaint and Raviart [4] made the first analysis of this method and proved a rate of convergence of $(\Delta x)^k$ for general triangulations and of $(\Delta x)^{k+1}$ for Cartesian grille. Later, Johnson and Pitkaranta [3] proved a rate of convergence $(\Delta x)^{k+1/2}$ for general triangulations and Peterson [7] numerically confirmed this rate to be optimal. Richter [9] obtained the optimal rate of convergence of $(\Delta x)^{k+1}$ for some structured two-dimensional non-Cartesian gride. In all the above papers, the exact solution is assumed to be very smooth. This assumption is hardly met in practice particularly when one deals with the equation of particle transport in domains with not smooth boundaries with discontinuer physical properties, eg.[15]. This motivated the author to develop a variational framework which could more easily account for the nonsmoothness of the solution of the transport equations and to analyse various categories of projectional methods for this equation [11,12]. The case in which the solution admits discontinuities was treated by Lin and Zhou [6] who proved the convergence of the method. The issue of the interrelation between the mesh and the order of convergence of the method was explored by Zhou and Lin [10], case $k = 1$, and later by Lin, Yan, and Zhou [5], case $k = 0$, and optimal error estimates were proven under suitable assumptions on the mesh. Recently, Falk and Richter [2] have obtained a rate of convergence of $(\Delta x)^{k+1/2}$ for general triangulations for Friedrich systems. Finally, Cockburn, Luskin, Shu, and Süli [1] have shown how to postprocess the approximate solution to obtain a rate of convergence of $(\Delta x)^{2k+1}$ in Cartesian grids.

Recent achievements in the field of discontinuous Galerkin methods and the issues of domain decomposition for the advection and diffusion equations has stimulated the author to extend his approach developed earlier so that a variety of local projectional methods solving the transport equation could be easily handed. The following sections provide an outline of the approach.

2. Weak solutions and approximate methods for the equations related to noncoercive bilinear forms

In recent years many modern techniques solving boundary value problems have been developed on basis of approximate variational formulations. This is usually done in terms of bilinear forms bounded in certain functional spaces. This technique easily provides us with the so called a priori estimate, which imply the uniqueness of the weak solutions. At the same time it is a convenient tool for investigation of the order of convergence of various approximate methods solving the original boundary value problem.

Consider now a formulations suitable for the transport equations.

Let and are two Hilbert spaces such that *B* is a dense subspace of and

(1) $\|u\|_L \le k\|u\|_B, \qquad u \in B$.

Thus we have

(2) $B \subset L \subset B^*, \quad k^{-1}\|u\|_{B^*} \le \|u\|_L \le k\|u\|_B$,

where B^* is dual of *B* with the point space *L*.

The norm in B^* can be defined by the formula

(3) $\|u\|_{B^*} = \sup_{v \notin B} \|\upsilon\|_B^{-1} |(u,\upsilon)|, \quad with (u,\upsilon)$,

being the inner product on *L*.

Consider the bilinear form $a(u,\upsilon)$ with the property

(4) $|a(a,\upsilon)| \le C_1\|u\|_L\|\upsilon\|_B$.

With the form a(u,v) we can associate the continuous linear operator *A* from *B* into *L* by the relation

(5) $(Au,\upsilon) = a(u,\upsilon), \quad \upsilon \in B$.

The domain of *A* is given by:

$$D(A) = \left\{ \begin{array}{l} u : for\ each\ \upsilon \notin B \\ there\ exists\ k_u < \infty\ such\ that \\ |a(u,\upsilon)| \le k_u\|\upsilon\|_L \end{array} \right\}.$$

We assume that the bilinear from $a(u,\upsilon)$ generates another form $a^*(u,\upsilon)$ such that

(6) $a^*(u,\upsilon) = \overline{a}(u,\upsilon)$ for $u,\upsilon \in B$,

and

(6a) $|a^*(u,\upsilon)| \le C\|u\|_L\|\upsilon\|_B$.

With the form $a^*(u,\upsilon)$ we can related an operator A^* in similar way as A to $a(u,\upsilon)$.

We require that *B=D(A₁)* and A_1 is a one-to-one transformation from *B* onto *L*. To ensure this property it is sufficient to assume $A_1 = A^*$ and

(7) $\text{Re}\, a(u,u) \ge C_2\|u\|_L^2$.

It is obvious that in this case

$$a(u,\upsilon) = (u, A^*\upsilon)$$

and

$$a^*(u,\upsilon) = (u, A\upsilon).$$

The Eq (7) can be considered as a generalized coerciveness condition for the form $a(u,\upsilon)$.

With this assumption valid we can prove [11] that for each $S \in B^*$ there exists a unique solution of the generalized variational problem.

<u>GVP</u>. Find $u \in L$ such that for each $\upsilon \in B$ we have

(8) $a(u,\upsilon) = S(\upsilon) \equiv (S,\upsilon)$.

Such a solution u is called the weak solution of the equation

(9) $Au = S$.

2. Projection methods

Consider two families of the spaces L_h and B_h, $h \in [0,1]$.

We define the approximate problem.

<u>AP.</u> Find $u_h \in L_h$ such that for each $\upsilon_h \in B_h$ we have

(10) $a(u_h,\upsilon_h) = S(\upsilon_h) = (S,\upsilon_h)$.

The set of assumptions concerning L_h and B_h relevant for further analysis we formulate as:

<u>Assumption A</u>

(i) For $h \to 0$, L_h and B_h tend to $\widetilde{L}$ and $\widetilde{B}$,which dense in *L* and *B*, respectively.

(ii) There exists $h_\upsilon \ge 0$ such that for $h \le h_o$ the projection P_h from L onto L_h is also a bijection of $\widetilde{L}_h$ onto L_h, where:

$$\widetilde{L}_h = A^* B_h.$$

(iii) $\tau = \lim_{h \to 0} \tau_h > 0$, where

$$\tau_h = \inf_{x_n \widetilde{L}_h, \|x_n\|=1} \|P_h x_h\|.$$

Under Assumption A for the problem AP we can prove the theorem concerning the convergence of u_h to the solution u of the problem GVP [1,2].

<u>Theorem 1.</u> There exists a unique solution of AP for each *h*. The sequence $\{u_h\}$ converges

weakly to the solution u of GVP. If moreover $L_h = \widetilde{L}_h$ then the sequence $\{u_h\}$ converges in the norm of L. The rate of convergence is determined by the best approximation of u by the elements of L_h.

Theorem 2. If

(i) the solution u of GVP belongs to

(ii) $L_h = B_h \subset B$

(iii) condition (7) is satisfied

then the solution u_h of AP problem are convergent in L norm to the solution u of GVP. The rate of convergence is given by

$$\|u - u_h\|_L \le \tag{11}$$
$$\le C \inf \inf_{\upsilon_h \notin B_h} \|u - \upsilon_h\| \inf_{\upsilon_h \notin B_h} \|\widetilde{u} - \upsilon_h\|,$$

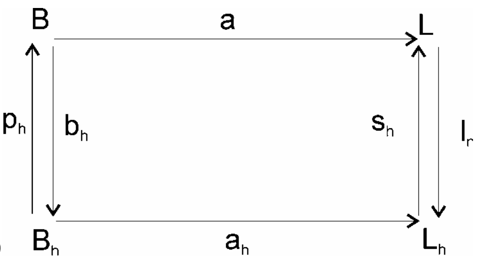
where $\widetilde{u}$ is the solution of

(11a) $a^*(\widetilde{u}, u) = (S^*, \upsilon), \; for\ each\ \upsilon \in B$

with

$$S^* = \frac{u - u_k}{\|u - u_h\|_L}.$$

Now, we can consider more general case when B_h and L_h are not subspace of B and L, respectively. That is we have the situation

(12) 

The operator A_h is defined by the form $a_h(u, \upsilon)$ satisfying the properties (4) and (6) stated in terms of the spaces B_h and L_h. The operators p_h, b_h and s_h and l_h make the "correspondence" among the elements of B and B_h and L conditions for the operators p_h, b_h and s_h, l_h we shall prove the generalizations of Theorems (1) and (2), stating the criteria of convergence in the norm of L and L_h spaces. Suppose now that $a_h(u_h, \upsilon_h)$ is of the form

(13) $a_h(u_h, \upsilon_h) = a(s_h u_h, p_h \upsilon_h)$.

Consider the approximate problem.

AP' Find $u_h \in L_h$ such that for each $\upsilon_h \in B_h$ we have

(14) $a_h(u_h, \upsilon_h) = (s_h, \upsilon_h)_h$,

where the bilinear form $a_h(u_h, \upsilon_h)$ is defined by Eq (13) and

(15) $(s_h, \upsilon_h)_h \equiv (S, p_h \upsilon_h)$.

Formulate the analog of Assumption A.

Assumption A'

(i) $s_h L_h$ and $p_h B_h$ are dense in the limit in L and B, respectively.

(ii) There exists $h_o \ge 0$ such that for $h \le h_o$ a projection P_h from L onto $s_h L_h$ is also a bijection of $\widetilde{L}_h$ onto $s_h L_s$, where $\widetilde{L}_h = A^* p_h B_h$

(iii) $\tau = \lim_{h \to 0} \tau_h > 0$, where

$$\tau_h = \inf_{z_n \in \widetilde{L}_h, \|z_n\| = 1} \|P_h z_h\|.$$

Making use of the technique developed in [11,12] and [13] we can prove the theorem.

Theorem 3. Under Assumption A' the results of Theorems 1 and 2 can be extended to problem AP', in particular

(i) $p_h u_h$ converges weakly to the solution u of GVP;

(ii) if $p_h L_h = \widetilde{L}_h = A^* p_h B_h$ then $\|u - p_h u_h\|_L \underset{k \to 0}{\rightarrow} 0$

(iii) if $u \in B, L_h = B_h$ and condition (6) in fulfilled, then

$$\|u - p_h u_h\|_L \le \tag{16}$$
$$\le C \inf_{\upsilon_h \in B_h} \|u - p_h \upsilon_h\|_B \inf_{\upsilon_h \in B_h} \|\widetilde{u} - p_h \upsilon_h\|_B,$$

where $\widetilde{u}$ is the solution of (11a) with

$$S^* = \frac{u - p_h u_h}{\|u - p_h u_h\|_L}.$$

Theorems 1-2 form the mathematical basis to analysis a certain class of approximate methods, therefore the finite element methods with approximating spaces L_h and B_h satisfying interelement continuity relations imposed by the properties the solutions to GVP. Theorem 3 is suitable for the standard finite difference approach to solve GVP.

It should be noticed that one can considerably weaken the requirement of interelement continuity for the function of B_h

and L_h and still obtain a convergent method. One gets so called nonconforming methods. They are based on the extended variational formulation, often called hybdride one, in which the continuity constraints are removed at the expense of introducing new terms in the bilinear form. In the following we formulate the hybrid method related to the bilinear forms satisfying Eqs (4) and (6) in suitable functional spaces. Generalization variations framework and local projection methods.

Suppose we have three Hilbert spaces $\overline{B}, \overline{L}$ and M. Let the bilinear form $a(u,\upsilon)$ satisfy the conditions analogous to (4) and (6). Let finally $b(\upsilon,\mu)$ be a bilinear form on $\overline{B}$ x M such that the quantity $\|\mu\|_*$ given by the equation:

(17) $$\|\mu\|_* = \sup_{\upsilon\in\overline{B}} \frac{b(\upsilon,\mu)}{\|\upsilon\|_{\overline{B}}}$$

is norm on M.

We define the hybrid variational problem.

<u>HVP</u> For given linear forms $f(\upsilon)$ and $g(\mu)$ continuous on $\overline{B}$ and M, respectively find a pair $(u,\lambda)\in\overline{L}xM$ such that for any $(\upsilon,\mu)\in\overline{B}xM$ we have

$$\overline{a}(u,\upsilon)+b(\upsilon,\lambda)=f(\upsilon)$$
$$b(u,\mu) \qquad = g(\mu).$$

In similar way we introduce the approximate problem <u>HAP</u> in terms of approximate forms

$$\overline{a}_h(u_h\upsilon_h), f_h(\upsilon_h), b(\upsilon_h,\mu_h)$$

and

$$g_h(\mu_h)$$

continuous on $\overline{L}_h x\overline{B}_h, \overline{B}_h, \overline{B}_h xM_h$ *and* M_h are dense in the limit in the corresponding spaces $\overline{B}, \overline{L}$ and M..

Following the reasoning of [11] and [14] we can prove the existence and uniqueness of the pair $(u,\lambda)\in\overline{L}xM$ which solves HVP. If $f(\upsilon)=(S,\upsilon)$ with $S\in\overline{L}$ then the solution (u,λ) belongs to $\overline{B}xM$. The similar statement is valid for a pair (u_h,λ_h) being a solution of <u>HAP</u>.

Now we estimate the error bounds for $\|u-u_h\|_{\overline{L}}$ and $\|\lambda-\lambda_h\|_M$. To do that we first define sets V_h and $\overline{V}_h$.

$$\overline{V}_h = \left\{ \begin{array}{l} \upsilon_h\in\overline{B}_h; b_h(\upsilon_h\mu_h)=0 \\ for \quad any \quad \mu_h\in M_h \end{array} \right\}$$

(18)

$$V_h = \left\{ \begin{array}{l} \upsilon_h\in\overline{B}_h; b_h(\upsilon_h,\mu_h)=g_h(\mu_h) \\ for \quad any \quad \mu_h\in M_h \end{array} \right\}$$

If $(u_h,\lambda_h)\in(\overline{B}_h,M_h)$ is a solution of HAP, then u_h solves the problem.

<u>HAP'</u> Find $u_h\in V_h$ such that for any $\upsilon_h\in V_h$

$$\overline{a}_h(u_h,\upsilon_h)=f_h(\upsilon_h)-b_h(\upsilon_h,\lambda_h).$$

<u>Lemma 1.</u> Sufficient conditions for the existence of $(u_h,\lambda_h)\in\overline{B}_h xM_h$ - the unique solution of HAP are:

(i) $\overline{L}_h=\overline{B}_h$;

(19) (ii) the form $\overline{a}_h(u_h\upsilon_h)$ satisfies Eqs (4) and (7);

(iii) $\|\mu_h\|_{*h} = \sup_{\upsilon_h\in\overline{B}_h} \frac{b_h(\upsilon_h,\mu_h)}{\|\upsilon_h\|_{\overline{B}}}$.

The proof of the lemma are based on that given in [14] modified in such a way that the generalized coerciveness condition (7) may be taken into account, [13].

The estimation of error bounds found in [13] for HAP are summarized in the following two theorems.

<u>Theorem 4.</u> Suppose that the assumptions of Lemma 1 are fulfilled, then there exists a unique solution $u_h\in\overline{B}_h$ of HAP'. The error $\|u-u_h\|_{\overline{L}}$ where *u* is the solution of HVP, can be estimated as follows:

$$\|u-u_h\|_{\overline{L}}^2 \le C$$
$$\Big\{ \inf_{\upsilon_h\in V_h}\left(\|u-\upsilon_h\|_{\overline{L}}\right)+\sup_{w_h\in V_h}\frac{|\overline{a}(\upsilon_h,w_h)-\overline{a}_h(\upsilon_h,w_h)|}{\|w_h\|_{\overline{B}}}$$
$$+\sup_{w_h\in V_h}\frac{|f(w_h)-f_h(w_h)|}{\|w_h\|_{\overline{B}}}+ \tag{20}$$
$$+\inf_{u_h\in M_h}\sup_{w_h\in V_h}\frac{|b(w_h,\lambda-u_h)|}{\|w_h\|_{\overline{B}}}+$$
$$+\sup_{w_h\in V_h}\frac{|b(w_n,u_n)-b_h(w_h,u_h)|}{\|w_h\|_{\overline{B}}} \Big\}$$

<u>Theorem 5.</u> We assume that the conditions of Lemma 1 are fulfilled. Then there exist a unique solution (u_h,λ_h) of HAP. In addition to the estimation of $\|u-u_h\|_{\overline{L}}$ stated in Theorem 4 we

have also for the term $\|\lambda-\lambda_h\|_M$ the following estimation.

$$\begin{aligned}(21)\quad &\|\lambda-\lambda_h\|_M \le C\\ &\{\ \|u-u_h\|_{\bar{L}} + \inf_{\upsilon_h\in \bar{B}_h}\Big(\ \|u-u_h\|_{\bar{L}} + \\ &+\sup_{w_h\in \bar{B}_h}\frac{|\bar{a}(\upsilon_h,w_h)-\bar{a}_h(\upsilon_h,w_h)|}{\|w_h\|_{\bar{B}}}\ \Big)+\\ &+\sup_{w_h\in \bar{B}_h}\frac{|f(w_h)-f_h(w_h)|}{\|w_h\|_{\bar{B}}}+\\ &+\inf_{u_h\in M_h}\{\ \|\lambda-u_h\|_M + \\ &+\sup_{w_h\in \bar{B}_h}\frac{|b(w_h,u_h)-b_h(w_h,u_h)|}{\|w_h\|_{\bar{B}}}\ \}\ \}\end{aligned}$$

The error bounds for $\|u-u_h\|_{\bar{L}}$ in Theorem 4 and a fortiori that of Theorem 5 for $\|\lambda-\lambda_h\|_M$ involve the quantity

$$(22)\quad \psi(\upsilon_h)=\|u-u_h\|_{\bar{L}} + \sup_{w_h\in V_h}\frac{|\bar{a}(\upsilon_h,w_h)-\bar{a}_h(\upsilon_h,w_h)|}{\|w_h\|_{\bar{B}}}$$

with $\upsilon_h\in \bar{V}_h$.

In practice to estimate $\psi(\upsilon_h)$ we must know interpolation properties of $\bar{V}_h$ with respect to a subset of $\bar{B}$ witch the solution u belongs to. The following theorem permits us to avoid such an inconvenience [13].

Theorem 6. Under the assumptions of Lemma 1 we have

$$\begin{aligned}(23)\quad &\inf_{\upsilon_h\in \bar{V}_h}\psi(\upsilon_h)\le \inf_{\upsilon_h\in \bar{B}_h}\psi(\upsilon_h)+\\ &+C\Big(\ \sup_{\mu_h\in M_h}\frac{|b(u-\upsilon_h)-\mu_h|}{\|\mu_h\|_M}+\\ &\sup_{\mu_h\in M_h}\frac{|b(\upsilon_h\mu_h)-b_h(\upsilon_h\mu_h)|}{\|\mu_h\|_M}\\ &+\sup_{\mu_h\in M_h}\frac{|g(\mu_h)-g_h(\mu_h)|}{\|\mu_h\|_M}\ \Big).\end{aligned}$$

Now we shall give an example how to relate HVP to GVP.

Suppose that the domain of the definition of functions being elements of the Hilbert spaces B and L used to formulate GVP is a convex set G in an Eucliden space. To denote that we can write $B=B(G)$ and $L=L(G)$. We can decompose G into disjoint sets G_i, $i=1,2,\ldots,n$, such that

$$(24)\qquad G=\bigcup_{i=1}^{n} G_i\,.$$

We introduce the product Hilbert spaces

$$(25)\quad \bar{B}=\prod_{i=1}^{n}B(G_i)\ \text{ and }\ \bar{L}=\prod_{i=1}^{n}L(G_i).$$

If the operator A related to the bilinear form $a(u,\upsilon)$ continuous in $L(G)xB(G)$ is an integrodifferential one, then any linear functional on $\bar{B}$ which vanishes on any $\upsilon\in B(G)$ can be represented in terms of a bilinear form $b(\upsilon,\mu)$ continuous in $\bar{B}xM$, that is

$$(26)\qquad F_\mu(\upsilon)=b(\upsilon,\mu),$$

where M is a suitable chosen Hilbert space of functions defined on $\bigcup_{i=1}^{n}\partial G_i$ where ∂G_i is boundary of G_i. In this case the hybrid variational problem HVP whose unique solution is also the solution of GVP can be defined by means of the forms $b(\upsilon,\mu)$ and $\bar{a}(u,\upsilon)$, where

$$(27)\qquad \bar{a}(u,\upsilon)=\sum_{i=1}^{n}a_i(u,\upsilon).$$

The formal definition of $a_i(u,\upsilon)$ in $L(G_i)xB(G_i)$ is the same as the form $a(u,\upsilon)$ of GVP in $L(G)xB(G)$.

3.Boundary value problem and approximate methods for the neutron transport equation

All the results of Sec.2 are directly applicable to analysis of various approximate method solving boundary value problem for the linear neutron transport equation. To see that we first introduce definitions useful for further analysis of the neutron transport equation.

We consider a subsed G of the six dimensional Euclidean space E. A point $\vec{x}$ of G will be represented by a triple: $\vec{x}=(\vec{r},\upsilon,\vec{\Omega})$, where $\vec{r}$ is a point of a convex set G_o in E^3, υ belongs to an interval $(0,\upsilon_M)$, and $\vec{\Omega}$ is a point of the unit sphere ω in E^3. The symbols can be interpreted as follows. The point $\vec{r}$ is a position

G_o where the neutron processes occur, $\vec{\upsilon} = \upsilon\vec{\Omega}$ denotes the neutron velocity and υ_M^2 is the maximal neutron energy.

Define the set Π_Ω to be an orthogonal projection of G_o on a plane perpendicular to $\vec{\Omega}$ and situated outside of G_o. With a fixed $\vec{\Omega} \in \omega$ and $\vec{r}_\Omega \in \Pi_{\vec{\Omega}}$ we associate the sets:

(28)
$$\Pi\left(\vec{r}_{\vec{\Omega}}, \vec{\Omega}\right) = G_o \cap \left\{\vec{r}_{\vec{\Omega}} + S\vec{\Omega}, s \in (-\infty, +\infty)\right\}.$$

The values of s corresponding to left and right end point of the interval $\Pi\left(\vec{r}_{\vec{\Omega}}, \vec{\Omega}\right)$ will be denoted by $s_1\left(\vec{r}_{\vec{\Omega}}, \vec{\Omega}\right)$ and $s_2\left(\vec{r}, \vec{\Omega}\right)$ respectively. The collection of all the points s_1 and s_2 induce the sets ∂G_+ and ∂G_-, where

(29)
$$\partial G_+ = \left\{\left(\vec{r}, \upsilon, \vec{\Omega}\right) \in \overline{G}, \vec{r}_{\vec{\Omega}} + s_2\vec{\Omega}\right\},$$
$$\partial G_- = \left\{\left(\vec{r}, \upsilon, \vec{\Omega}\right) \in \overline{G}, \vec{r} = \vec{r}_{\vec{\Omega}} + s_1\vec{\Omega}\right\}.$$

How we consider the problem of neutron transport. The density of neutrons $\psi\left(\vec{r}, \upsilon, \vec{\Omega}\right)$ in G due to a descributed neutron source $Q\left(\vec{r}, \upsilon, \vec{\Omega}\right)$ is a solution of the neutron transport equation

(30)
$$(T + \upsilon\Sigma - K)\Psi\left(\vec{r}, \upsilon, \vec{\Omega}\right) = Q\left(\vec{r}, \upsilon, \vec{\Omega}\right), \quad \left(\vec{r}, \upsilon, \vec{\Omega}\right) \in G$$

with the boundary condition

(31)
$$(\gamma_- \Psi)\left(\vec{r}_p, \upsilon, \vec{\Omega}\right) \equiv \Psi\left(\vec{r}_p, \upsilon, \vec{\Omega}\right) = \eta\left(r_p, \upsilon, \vec{\Omega}\right)\left(\vec{r}_p, \upsilon, \vec{\Omega}\right) \in \partial G_-.$$

The symbols in Eq.(30) are defined as follows:

(32)
$$(T\Psi)\left(\vec{r}, \upsilon, \vec{\Omega}\right) = \vec{\Omega}\upsilon\, grad_{\vec{r}}\, \Psi\left(\vec{r}, \upsilon, \vec{\Omega}\right),$$
$$(\upsilon\Sigma\Psi)\left(\vec{r}, \upsilon, \vec{\Omega}\right) = \upsilon\Sigma_t(\vec{r}, \vec{\upsilon})\Psi\left(\vec{r}, \upsilon, \vec{\Omega}\right),$$
$$(K\Psi)\left(\vec{r}, \upsilon, \vec{\Omega}\right) = \int_0^{\upsilon_M} \upsilon'^2 d\upsilon' \int_\omega d\vec{\Omega}' K\left(\vec{r}, \upsilon, \upsilon', \vec{\Omega}, \vec{\Omega}'\right) \cdot \Psi\left(\vec{r}, \upsilon, \vec{\Omega}\right)$$

where $\Sigma_t(\vec{r}, \upsilon)$ and $K\left(\vec{r}, \upsilon, \upsilon', \vec{\Omega}, \vec{\Omega}'\right)$ have standard meaning.

We denote by $L_p^i(X)$, i=0,1,2,... $p \geq 1$ the space of on a set X. In general by the symbol $Y^*(X)$ we shall understand the space dual to $Y(X)$. The dual product on $Y(X)xY^*(X)$ will be denoted by $(\ ,\ )$ if $X = G$, and by the symbols $\langle\ ,\ \rangle_+$ and $\langle\ ,\ \rangle_-$ for $X = \partial G_-$ and $X = \partial G_-$ respectively. Let B_p be the subspace of $L_p^0(G)$ consisting of functions Ψ such that

(33)
$$\begin{cases} (i) & \Psi \in L_p^0(G) \cap L_p^1(G) \\ (ii) & \textit{for almost all } \vec{r}_{\vec{\Omega}} \in \Pi_{\vec{\Omega}}, \vec{\Omega} \in \omega, \upsilon \in (0, \upsilon_M) \\ & \textit{is absolutely continuous on } \Pi\left(\vec{r}_{\vec{\Omega}}, \vec{\Omega}\right) \\ (iii) & T \in L_p^0(G) \textit{ for all } \vec{r}_{\vec{\Omega}}, \vec{\Omega} \textit{ and } \upsilon \textit{ satysfying } (ii) \\ (iv) & \gamma_+ \Psi\left(\vec{r}, \upsilon, \vec{\Omega}\right) \equiv \Psi\left(\vec{r}_p, \upsilon, \vec{\Omega}\right), \left(\vec{r}_p, \upsilon, \vec{\Omega}\right) \in \partial G_+ \\ & \textit{exists and belongs to } L_p^1(\partial G_+) \\ (v) & \|\Psi\|_{Bp} = \|\Psi\|_{L_p^0(G)} + \|T\Psi\|_{L_p^1(\partial G_+)} + \|\gamma_- \Psi\|_{L_p^1(\partial G_-)} \end{cases}$$

The properties of B_2 have been extensively studied in [11]. In [11] and [12] it is proved that the operator $A = (T + \upsilon\Sigma - K) \cdot \gamma_+$ is bijection from B_p onto $L_{p_-} \equiv L_p^0(G) \cdot L_p^1(\partial G_-)$ provided zero does not belong to the spectrum of $\widetilde{A} = T + \upsilon\Sigma - K$ with $D(\widetilde{A}) = B_p^0 \equiv \{\Psi \in B_p, \gamma\Psi = 0\}$. Moreover we have

(34)
$$\left\|A^{-1}\right\|_{L_{p_-} \to B_p} \leq C \leq \infty.$$

Similar conclusions are valid for the operator

$$A^* = \left(T + \upsilon\Sigma - K^*\right) \cdot \gamma_-.$$

In the theorems ensuring (34) presented in [11] and [12] singular slowing down kernels and $\upsilon_M = \infty$ were admitted. Define the form $a(u, w)$ bilinear on $L_{p_+} \cdot B_q$ by the formula

(35)
$$a(u, w) = \left(u, A^* w\right)_L \equiv (\Psi, -T_w) + \left(\Psi, \left(\upsilon\Sigma - K^*\right)w\right) + \left\langle \upsilon\Psi_p, \gamma_+, w\right\rangle$$

where $u = \{\Psi, \Psi_p\} \in L_{p_+} = L_p^0(G) \cdot L_p^1(\partial G_+)$ and $w \in B_q$.

Take $\{Q, \eta\} \in B_q^*$ then by the results of [11] there exists a unique solution $u = \{\Psi, \Psi_p\}$ of GVP defined by Eq. (8) with

(35a)
$$S(w) = (Q, w) + \left\langle \upsilon_\eta, \gamma_-, w\right\rangle_-.$$

For u we have

(36) $\left\| \{\Psi, \Psi_p\} \right\|_{L_{p+}} \le c \left\| \{Q, \eta\} \right\|_{B_q^*}$,

where c is the constant of Eq. (31). If $\{Q, \eta\} \in L_{p\_}$ then

$$\Psi \in B_p$$

(37) $\gamma_+ \Psi = \Psi_p \in L_p^1(\partial G_+)$,

$$\|\Psi\|_{B_p} \le c\{ \|Q\|_{L_p^0(G)} + \|\eta\|_{L_p^0(\partial G_+)} \}.$$

It should be noted that $a(u, \upsilon)$ defined by Eq. (32) is cohesive in $L_{2+} = L_2^0(G) \cdot L_2^1(\partial G_+)$ if the following condition is satisfied

(38) $((\upsilon\Sigma - K)\Psi, \Psi) \ge \gamma \|\Psi\|_{L_2^0(G)}^2$.

Similar results are valid for the GVP related to the form

$$a^*(u, w) = (\Psi, Tw) + (\Psi, (\upsilon\Sigma - K)w) + \\ + \langle \upsilon\Psi_p, \gamma_- w \rangle_-$$

with $u = \{\Psi, \Psi_p\} \in L_p$,

$$w \in B_q.$$

It is easy to see that there is the complete correspondence among the spaces B_q and $L_{p\pm}$, used in the definition of GVP for the neutron transport equation, and the spaces B and L from the previous sections where the general theory of approximate methods is presented. Therefore all the estimations of error bounds stead in Theorems 1 and 3 are valid for projectional and finite difference methods solving boundary value problem for the neutron transport equation. In particular they are applicable to

(i) spherical harmonics method,
(ii) finite element method,
(iii) general Galerkin method.

Suppose that the set G_o of position vectors $\vec{r}$ is partitioned into $G_i, i = 1, ..., N$ disjoint subregions with the boundaries ∂G_i

(39) $G_0 = \bigcup_{i=1}^{N} G_i$.

The form $b(u, \mu)$ bilinear on BxM suitable for HVQ for the neutron transport equation is defined as follows

$$b(u, \mu) =$$

(40) $= \sum_{i=1}^{N} \left(\langle \upsilon\gamma_+ u_i, \mu_i^+ \rangle_{i+} - \langle \tau\gamma_- u_i, \mu_i^- \rangle_{i-} \right)$

where

(40a) $u = \{u_i\} \in \overline{B} = \prod_{i=1}^{N} B_p(G_i)$;

$$\mu = \{\mu_i^\pm\} \in M = \prod_{i=1}^{N} L^0(\partial G_{i\pm}).$$

In similar way we define $b_h(u_h, \mu_h), \overline{B}_h$ and M_h. The elements of $B_p(G_i)$, $L^0(\partial G_+)$ and $L^0(\partial G_-)$ can be interpreted as the neutron distribution in G_i, outcoming and incoming partial interface currents respectively.

With $b(u, \mu)$ defined by Eq. (40) and the bilinear form $\overline{a}(u, \mu)$ given by Eq. (35), according to the recipe of Sec. 3 we can analyze in frame of HVP the following approximate methods solving the neutron transport equation

(i) variational formulation with discontinuous in space variable trial and weight functions,
(ii) partial boundary current method,
(iii) response matrix method,
(iv) local Green function method.
(v) local discontinuous Galerkin method.

4. Conclusion

This local projections methods are better suited than finite difference methods to handle complicated geometries. Second, the method can easily handle adaptivity strategies since the refining or unrefining of the grid can be done without taking into account the continuity restrictions type of conforming finite element methods. Also, the degree of the approximating polynomial can be easily changed from one element to the other. Adaptivity is of particular importance in hyperbolic problems given the complexity of the structure of the discontinuities. Third, the method is highly parallelizable. Since the elements are discontinuous, the mass matrix is block diagonal and since the order of the blocks is equal to the number of degrees of freedom inside the corresponding elements, the blocks can be inverted by hand once and for all.

The variational framework outlined in the previous sections is useful for introducing generalized notion of solutions to boundary value problems of the transport equation and to analyse various approaches to domain decompositions and convergence of local projectional methods.

References:

[1] B. Cockburn, M. Luskin, C.-W. Shu, and E. Süli. *A priori error estimates for the discontinuous Galerkin method. in preparation*.

[2] R.S. Falk and G.R. Richter. *Explicit finite element methods for symmetric hyperbolic equations*. SIAM J. Numer. Anal., 36(3), 935-952, 1999.

[3] C. Johnson and J. Pitkaranta. *An analysis of the discontinuous Galerkin method for a scalar hyperbolic equation.* Math. Comp., 46:1-26, 1986.

[4] P. LeSaint and P.A. Raviart. *On a finite element methods for solving the neutron transport equation.* In C. de Boor, editor, Mathematical aspects of finite elements in partial differential equations, pages 89-145. Academic Press, 1974.

[5] Q. Lin, N. Yan, and A.-H. Zhou. *An optimal error estimate of the discontinuous Galerkin method.* Journal of Engineering Mathematics, 13:101-105, 1996.

[6] Q,. Lin and A.-H. Zhou. *Convergence of the discontinuous Galerkin method for a scalar hyperbolic equation.* Acta Math.. Sci., 13:207-210, 1993.

[7] T. Peterson. *A note on the convergence of the discontinuous Galerkin method for a scalar hyperbolic equation.* SIAM J. Numer. Anal., 28:133-140, 1991.

[8] W.H. Reed and T.R. Hill. *Triangular mesh methods for the neutron transport equation.* Technical Report LA-UR-73-479, Los Alamos Scientific Laboratory, 1973.

[9] G.R. Richter. *An optimal-order error estimate for the discontinuous Galerkin method.* Math. Comp., 50:75-88, 1988.

[10] A.-H. Zhou and Q. Lin. *Optimal and superconvergence estimates of the finite element method for a scalar hyperbolic equation.* Acta Math. Sci., 14:90-94, 1994.

[11] M.Borysiewicz, R.Stankiewicz, *Weak Solution and Approximate Methods for the transport equation* Jour.Math.Anal.Appl., 68,191, 1979.

[12] M.Borysiewicz,R.Stankiewicz, *Variantional Formulation and Projectional Methods for the Second Order Transport Equation*, Jour.Math.Anal.Appl., 71, 210, 1979.

[13] M.Borysiewicz, G.Spiga , *Approximate Methods Solving Variational Problems Generated by Noncoercive Forms*, Progress Report, IAEA Contract 1236/RB, 1978.

[14] J.M.Thomas, *Sur l'Analyse Numerique des Methods d'Elements Finis Hybrides et Mixtes* These Universite Pierre et Marie Curie, Paris, 1977.

[15] M.Borysiewicz, N. Kruszyńska *Smootness of the solution of the 2D neutron transport equation in the polygon region*, Rap. 0-322/CYF/78, IBJ, Świerk, 1978.