On generalized periodic matrices*

CARMEN COLL AND NÉSTOR THOME
Departamento de Matemática Aplicada
Universidad Politécnica de Valencia,
46071 Valencia
SPAIN

Abstract:- In this paper, $\{\alpha$, k$\}$-group periodic matrices are introduced and analyzed. Using different approach (as $\{1\}$-inverses, Moore-Penrose inverse, elementary matrices, left and right inverses and singular value decomposition) we characterize this kind of matrices. Some results presented are an extension of those known about group inverse and group periodic matrices.

Key–Words:- Generalized inverse, group involutory matrix, finite Markov chain, singular system.

1 Introduction

It is well-known that a matrix $A \in \mathbb{C}^{n\times n}$ is said to be *involutory* if $A^{-1} = A$ and *periodic* if $A^k = I_n$, for some $k \in \mathbb{N}$, where I_n denotes the $n \times n$ identity matrix.

In 1998, group involutory matrices (that is, square matrices which coincide with its group inverse) were introduced and characterized in [4].

In [2], the authors have introduced group periodic matrices and furthermore they gave some characterizations of this kind of matrices.

In 2001, Drazin involutory matrices and Moore-Penrose involutory matrices were introduced and characterized in [8], together with another characterization of group involutory matrices.

In this work, $\{\alpha, k\}$-group periodic matrices will be studied. This new kind of matrices is an extension of those presented in [2, 4, 8] and they permit to say when a group inverse is a monomial term. The motivation of this work arises from the fact that the group inverse of a matrix $A \in \mathbb{C}^{n\times n}$ is a polinomial in A (see [1]).

Some neccesary and sufficient conditions are developed from different points of view. That is, we will characterize $\{\alpha, k\}$-group periodic matrices in terms of different representations of the group inverse of a square matrix.

Some obtained results about group inverse or Drazin inverse can be applied to give the general solution of singular control time-invariant systems (see [5, 9]) or they can be used to give the solution of a subclass of them as for instance: symmetric singular control systems (see [3]).

* This work was partially supported by Spanish grant BFM2001-2783.

2 $\{\alpha, k\}$-Group Periodic Matrices

We recall that the *group inverse* of a square matrix $A \in \mathbb{C}^{n\times n}$ is the unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying $AXA = A$, $XAX = X$ and $AX = XA$ and it is denoted by $A^{\#}$. It is well-known that $A^{\#}$ there exists if and only if $\text{rank}(A^2) = \text{rank}(A)$ (see [1]).

We start this section with the following definition which indicates when a group inverse matrix coincides with a monomial term.

Definition 1 *Let $A \in \mathbb{C}^{n\times n}$, k be an integrer number, $k \geq 2$, and $\alpha \in \mathbb{R}$. The matrix A is said to be $\{\alpha,\ k\}$-group periodic matrix if $A^{\#} = \alpha A^{k-1}$.*

In order to ilustrate the definition we give the following example.

Example 1 *The group inverse of the matrix*

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

It is easy to see by induction that its powers are

$$A^k = \begin{bmatrix} 1 & -k & 0 \\ 0 & 1 & 0 \\ 1 & 1-k & 0 \end{bmatrix}, \ k \geq 1.$$

Comparing the expressions for $A^{\#}$ and A^k, we deduce that there not exist an integer $k \geq 2$ and a real α such that A is $\{\alpha, k\}$-group periodic matrix.

Example 2 *It is easy to see that the matrix*

$$B = \begin{bmatrix} 0 & -1 \\ 0 & -i \end{bmatrix} \in \mathbb{C}^{2\times 2}$$

is $\{1,3\}$-group involutory matrix.

Observe that if $k = 2$ and $\alpha = 1$, this concept coincides with the particular case of group involutory matrices studied in [4] and if $\alpha = 1$ and $k > 2$, it coincides with the group periodic matrices studied in [2].

We will stand for $\mathcal{G}_1(A)$ the set of all $\{1\}$-inverses of A, that is matrices A^- such that $AA^-A = A$ and for $A^\dagger$ the Moore-Penrose inverse of A (see [1]). Moreover, $Q_A = I_n - AA^\dagger$, $Q_{A^*} = I_n - A^*(A^*)^\dagger$, N_R is a right inverse of N and M_L is a left inverse of M. The remainder notation used in this paper can be found in [4].

In the following result we present several equivalent conditions to that given in the definition 1.

Theorem 1 *Let $A \in \mathbb{C}^{n\times n}$, $k \in \mathbb{N}$, $k \geq 2$. Then the following conditions are equivalent.*

(a) A is a $\{\alpha,\ k\}$-group periodic matrix,

(b) $\alpha A^{k-1} = A^-AA^-$, where $A^- \in \mathcal{G}_1(A)$ satisfies $AA^- = A^-A$,

(c) $\alpha A^{k-1} = \mathcal{A}A\mathcal{A}$, where

$$\mathcal{A} = (A^\dagger + W_1 Q_A + Q_{A^*} W_2),$$

and $W_1, W_2 \in \mathbb{C}^{n\times n}$ satisfy

$$AA^\dagger + AW_1Q_A = A^\dagger A + Q_{A^*}W_2A,$$

(d) $\alpha A^{k-1} = A^- + X - A^-AXAA^-$, where X satisfies

$$\begin{aligned} &A^-AA^- + A^-AX - A^-AXAA^- \\ &+XAA^- + XAX = \\ &= XAXAA^- + A^-AXAX \\ &-A^-AXAXAA^- + A^- + X, \end{aligned}$$

and

$$AA^- + AX(I - AA^-) = A^-A + (I - A^-A)XA,$$

(e) if $W = (Q_1P_1A_1)^{k-1}Q_1$ then

$$\begin{bmatrix} A_1^{-1} & X \\ Y & YA_1X \end{bmatrix} = \alpha \begin{bmatrix} WP_1 & W_1P_2 \\ ZQ_1P_1 & ZQ_1P_2 \end{bmatrix},$$

where X, Y satisfy

$$QP \begin{bmatrix} I & A_1X \\ O & O \end{bmatrix} = \begin{bmatrix} I & O \\ YA_1 & O \end{bmatrix} QP,$$

and

$$Z = \begin{cases} Z_1 & \text{if } k \text{ is odd} \\ Z_2 & \text{if } k \text{ is even} \end{cases}$$

$$Z_1 = \prod_{i=1}^{(k-1)/2} Q_2P_AQ_1P_A$$

$$Z_2 = \left[\prod_{i=1}^{(k-2)/2} Q_2P_AQ_1P_A\right] Q_2P_A$$

where $P_A = P_1A_1$,

(f) $\alpha A^{k-1} = N_R^{-1}M_L^{-1}$, where $MM_L^{-1} = N_R^{-1}N$,

(g) if X, Y satisfy

$$U_1U_1^* + U_1\Sigma XU_2^* = V_1V_1^* + V_2Y\Sigma V_1^*,$$

then

$$\alpha(U_1\Sigma V_1^*)^{k-1} = [V_1\Sigma^{-1} + V_2Y][U_1^* + \Sigma XU_2^*]$$

(h) the series $X + (I - XA)X + (I - XA)^2X + \cdots$ converges to the matrix αA^{k-1}, if conditions (a), (b), (c) and (d) of theorem 6 of [4] hold.

Sketch of the proof: Since the complete prove is so long, we only present a sketch of it.

Different representations of a group inverse are used to prove the equivalences (a)-(g). Furthermore, to establish the equivalence between the conditions (a) and (e) it is neccesary to show (by induction) that

$$\left(QP\begin{bmatrix} I & A_1X \\ O & O \end{bmatrix}\right)^{k-1} QP =$$

$$\begin{bmatrix} (Q_1P_A)^{k-1}Q_1P_1 & (Q_1P_A)^{k-1}Q_1P_2 \\ ZQ_1P_1 & ZQ_1P_2 \end{bmatrix},$$

where $P_A = P_1A_1$.

In the same way, to establish the equivalence between the conditions (a) and (g) it is neccesary to show (by induction) the following relation

$$\left(U\begin{bmatrix} \Sigma & O \\ O & O \end{bmatrix}V^*\right)^{k-1} = (U_1\Sigma V_1^*)^{k-1}.$$

This complete the proof.

3 Some applications

Firstly, we give an application related to solution of singular systems.

Consider an autonomous singular system

$$Ex\,(k+1) = Ax\,(k)\,.$$

If there exist $\lambda \in \mathbb{C}$, such that $\det(\lambda E - A) \neq 0$, then the system has a solution using the matrices $\widehat{E} = (\lambda E - A)^{-1} E$ and $\widehat{A} = (\lambda E - A)^{-1} A$. The solution of this system can be easly computed when $\widehat{E}$ is an $\{\alpha, k_0\}$-group periodic matrix, and this solution is given by

$$x\,(k) = \alpha^k \widehat{E}^{(k_0-1)k}\widehat{A}^k x\,(0)\,,\ \ k \geq 1.$$

Note that, a direct property of an $\{\alpha, k\}$-group periodic matrix is that, $E = \alpha E^{k+1}$. So, if consider $\left\{E^k\right\}_{k=0}^{\infty}$, with E an $\{\alpha, k\}$-group periodic matrix, we only have a finite number of different powers in the sequence. This fact can be used to calculate the above solution.

On the other hand, the finite Markov chains are one of the most interesting applications of the generalized inverses. In this theory the stochastic matrices play an important role. We remain that a stochastic matrix is a matrix

$$P = [p_{ij}]_{i,j=1}^n$$

such that $0 \leq p_{ij} \leq 1$ and

$$\sum_{i=1}^n p_{ij} = 1,\ \ j = 1, 2, \ldots, n.$$

It is well-known that if P is a stochastic matrix and if $A = I - P$, then $\mathrm{ind}(A) = 1$, see [6]. This fact does that the group inverse can be used in some problems involving stochastic matrices.

Furthemore, in the study of Markov chain, frequently appears the matrix $I - AA^{\#}$, because this matrix is the result of different limiting processes. For instance, see [6],

$$\lim_{n\to\infty} \frac{I + P + \cdots + P^{n-1}}{n} = I - AA^{\#}$$

where P is a stochastic matrix and $A = I - P$.

If our matrices are $\{\alpha, k\}$-group periodic matrices, it is clear that above calculation can be reduced.

We clarify these comments with one example.

Example 3 *If the stochastic matrix P is given by*

$$P = \begin{bmatrix} 0.5 & 0 & 0 \\ 0.5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

then

$$A = I - P = \begin{bmatrix} 0.5 & 0 & 0 \\ -0.5 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Note that, the matrix A is a $\{4, 2\}$-group periodic matrix, since $A^{\#} = 4A$. Then,

$$\lim_{n\to\infty} \frac{I + P + \cdots + P^{n-1}}{n} = I - 4A^2$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

4 Conclusions

In this work, we introduced $\{\alpha, k\}$-group periodic matrices as an extension of those given in [2] and [4]. We have found neccesary and sufficient conditions which characterize these new kind of matrices. Different factorizations of the group inverse (involving $\{1\}$-inverses, Moore-Penrose inverse, elementary matrices, left and right inverses and singular value decomposition) have been used to give characterizations of the $\{\alpha, k\}$-group periodic matrices.

References:

[1] *A. Ben-Israel* and *T. Greville*, *Generalized inverses: theory and applications*, John Wiley and Sons, New York, 1974.

[2] *R. Bru, C. Coll* and *N. Thome*, Group Periodic Matrices. *Recent Advances in Applied Applied and Theoretical Mathematics: A series of Reference Books and Textbooks,* 13–16, ISBN: 960-8052-21-1, Editor: N. Mastorakis, Grecia, 2000.

[3] *R. Bru, C. Coll* and *N. Thome,* Symmetric Singular Linear Control Systems, *Applied Mathematics Letters, 2001.*

[4] *R. Bru* and *N. Thome,* Group Inverse and Group Involutory Matrices, *Linear and Multilinear Algebra,*Vol 45, 1998, 207-218.

[5] *S. L. Campbell, Singular systems of differential equations,* Pitman Advanced Publishing Program, San Francisco, 1980.

[6] *S. L. Campbell* and *C. D. Meyer, Generalized inverses of linear transformation*, Dover Publications, New York, 1979.

[7] *S. L. Campbell, C. D. Meyer, Jr.* and *N. J. Rose,* Applications of the Drazin inverse to linear systems of differential equations, *SIAM J. Appl. Math.,* Vol. 31, 1976, pp. 411–425.

[8] *J. Climent, N. Thome* and *Y. Wei,* Geometrical Approach on Generalized Inverse Matrices. *Linear Algebra and its Applications*, Vol. 332–334, 535–542, 2001.

[9] *T. Kaczorek*, *Linear Control Systems, Analysis of Multivariable Systems.* John Wiley & Sons Inc., Vol. I, New York, 1992