Hilbert Matrix Operator on weighted sequence spaces

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Abstract: *In this paper we extend our work, finding the norm of the Hilbert Matrix operator,* to the weighting sequence spaces $d(w,p)$, with the weighting sequences $w = (w_n)$ defind either $w_n = 1/n^\alpha$ or $W_n = n^{1-\alpha}$.

Key Words: *Hilbert operator, norm and Banach sequence spaces.*

1. Introduction

In [Lash 1,2,3], the author determined the norms and lower bounds of the Hilbert, Copson and averaging operators on the Lorentz sequence space $d(w,1)$, with the weighting sequence (w_n) defined either by $w_n = 1/n^\alpha$ by or $W_n = n^{1-\alpha}$, where $W_n = w_1 + \cdots + w_n$ (the second choice arises as "naturally" as the first in the context of Lorentz spaces). In the present paper, we address the problem of finding the norm of the Hilbert operator in the case $p > 1$. The problem of lower bounds was considered in [JL].

We consider one version of the Hilbert operator (which we denote by H). The classical inequality of Hilbert describe the norm of this operator on ℓ_p (where $p > 1$). Solutions to our problem need to reproduce this inequality when we take $w_n = 1$, and the results of [Lash 1,2,3] when we take $p = 1$. The norms of this operator on $d(w,p)$ are determined by their action on *decreasing*, non-negative sequences. In the case $p = 1$, the norms are already determined by the elements $e_1 + \cdots + e_n$. This is no longer true when $p > 1$, and consequently the methods of [Lash 2,3] no longer apply. Some estimations for norms of operators on $\ell_p(w)$ have been given in [AH] and [Benn 1,2]. These take the form of equivalent expressions in terms of the weighting sequences. However, they do not really help with the specific problems considered here: typically, they simply transfer the problem to evaluation of other, equally difficult, upper bounds.

The most closely analogous problem for the continuous case is to find the norms of the corresponding operators on the weighted functions spaces$L_p(w)$ (rather than Lorentz function spaces), restricted to decreasing functions. There is an extensive literature on such operators(e,g. [AH], [AM], [HK], [Muck]). Our two special choices of w are alternative analogues of the weighting function $1/x^\alpha$,and the continuous analogous of our problem have solutions that are either known explicitly or follow easily from known results. For the Hilbert operator H, the bilinear method can be adapted to show that the value from the continuous case is reproduced: for either choice of w,

$$\|H\|_{w,p} = \frac{\pi}{\sin[(1-\alpha)\pi/p]}.$$

2. Preliminaries

For a sequence $x = (x_n)$, we define $|x|$ and the relation $x \leq y$ in the obvious way. We denote by e_j the sequence having 1 in place j and 0 elsewhere. Let $w = (w_n)$ be a decreasing, non-negative sequence with $\lim_{n\to\infty} w_n = 0$ and $\sum_{n=1}^{\infty} w_n$ is divergent. The Lorentz sequence space $d(w,p)$ is then defined as follows. Given a null sequence $x = (x_n)$, let (x_n^*) be the decreasing rearrangement of $|x_n|$. Then $d(w,p)$ is the space of null sequences x with norm

$$\|x\|_{w,p} = \left(\sum_{n=1}^{\infty} w_n {x_n^*}^p\right)^{1/p}.$$

Now consider the operator T defined by $Tx = y$, where $y_i = \sum_{n=1}^{\infty} t_{i,j}x_j$. We denote $\|T\|_{w,p}$ the norm of T as an operator from $d(w,p)$ to itself. We assume throughout that $t_{i,j} \geq 0$ for all i,j, which implies in each case that the norm is determined by the action of T on non-negative sequences. In the next lemma, we establish conditions, adequate for the operators considered below, guaranteeing that $\|T\|_{w,p}$ is determined by *decreasing*, non-negative sequences (more general conditions are given in [Lash 1], Proposition 1.4.1).

Lemma 2.1. Suppose that T, given by the matrix $(t_{i,j})$, maps $d(w,p)$ into $d(w,p)$. Write $c_{m,j} = \sum_{i=1}^{m} t_{i,j}$. Suppose that:

(i) $t_{i,j} \geq 0$ for all i,j;

(ii) $\lim_{j\to\infty} t_{i,j} = 0$ for each i; and either

(iii) $t_{i,j}$ decreases with j for each i, or

(iv) $t_{i,j}$ decreases with i for each j and $c_{m,j}$ decreases with j for each m.

Then $\|T(x^*)\|_{w,p} \geq \|T(x)\|_{w,p}$ for all non-negative elements x of $d(w,p)$.

Proof. Let $y = T(x)$ and $z = T(x^*)$. As before, write $X_j = x_1 + \cdots + x_j$, etc. First, assume condition (iii). By Abel summation (which is valid because of condition (ii)), we have

$$y_i = \sum_{j=1}^{\infty} t_{i,j}x_j = \sum_{j=1}^{\infty}(t_{i,j} - t_{i,j+1})X_j,$$

and similarly for z_i with X_j^* replacing X_j. Since $X_j \leq X_j^*$ for all j, we have $y_i \leq z_i$ for all i, which implies that $\|y\|_{w,p} \leq \|z\|_{w,p}$. Now assume (iv). Then y_i and z_i decrease with i, and

$$\begin{aligned} Y_m = \sum_{i=1}^{m}\sum_{j=1}^{\infty} a_{i,j}x_j &= \sum_{j=1}^{\infty} c_{m,j}x_j \\ &= \sum_{j=1}^{\infty}(c_{m,j} - c_{m,j+1})X_j \end{aligned}$$

and similarly for Z_m. Hence $Y_m \leq Z_m$ for all m. By the majorization principle (also known as Karamata's inequality) (e.g. [BB, 1.30]), this implies that $\sum_{i=1}^{m} y_i^p \leq \sum_{i=1}^{m} z_i^p$ for all m, and hence by Abel summation that $\|y\|_{w,p} \leq \|z\|_{w,p}$. □

Under the condition of Lemma 2.1, it will be sufficient to consider the action of T on *decreasing,* non-negative sequences in $d(w,p)$. Evaluations in [Lash 1, 3] are based on the property, specially for $p = 1$, that $\|T\|_{w,p}$ is determined by the elements $e_1 + \cdots + e_n$. These statements fail when $p > 1$ (with or without weights). The next example shows, in the setting of Hardy's inequality, this is not true when $p > 1$. Our objective is to determine $\|T\|_{w,p}$ for the Hilbert operator. *The "bilinear" method.* The continuous analogues of the problem considered here can all be solved in a straightforward way by this method (cf. [HLP], sections 9.2 and 9.3). Here we describe a slightly generalized form of the method for the discrete case.

Lemma 2.2. Let $p > 1$ and $p^* = p/(p-1)$. Let A be the operator with matrix $(a_{i,j})$, where $a_{i,j} \geq 0$ for all i,j. Suppose that (s_i), (t_j) are two sequences of strictly positive numbers such that for some K_1, K_2:

$$s_i^{1/p}\sum_{j=1}^{\infty} a_{i,j}t_j^{-1/p} \leq K_1 \quad \text{for all i},$$

$$t_j^{1/p^*}\sum_{i=1}^{\infty} a_{i,j}s_i^{-1/p^*} \leq K_2 \quad \text{for all j}.$$

Then for all non-negative sequences $x \in \ell_{p^*}$ and $y \in \ell_p$,

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} x_i a_{i,j} y_j \leq K_1^{1/p^*}K_2^{1/p}\|x\|_{p^*}\|y\|_p,$$

hence $\|A\|_p \leq K_1^{1/p^*}K_2^{1/p}$.

Proof. We have $x_i a_{i,j} y_j = c_{i,j}d_{i,j}$, where

$$c_{i,j} = x_i\, a_{i,j}^{1/p^*} s_i^{1/pp^*} t_j^{-1/pp^*},$$

$$d_{i,j} = y_j\, a_{i,j}^{1/p} s_i^{-1/pp^*} t_j^{1/pp^*}.$$

By Hölder's inequality (applied to the double sum),

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} c_{i,j}d_{i,j} \leq C^{1/p^*}D^{1/p}$$

, where

$$C = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} c_{i,j}^{p^*} = \sum_{i=1}^{\infty} x_i^{p^*} s_i^{1/p}\sum_{j=1}^{\infty} a_{i,j}t_j^{-1/p} \leq K_1\sum_{i=1}^{\infty} x_i^{p^*},$$

$$D=\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}d_{i,j}^{p}=\sum_{j=1}^{\infty}y_j^p t_j^{1/p^*}\sum_{i=1}^{\infty}a_{i,j}s_i^{-1/p^*}$$

$$\leq K_2\sum_{j=1}^{\infty}y_j^p.$$

The last statement follows by the duality of ℓ_p and ℓ_{p^*}. □

In many cases, this is applied with $s_j = t_j = j$. We shall be particularly concerned with two choices of w, defined respectively by $w_n = 1/n^\alpha$ (where $0 \leq \alpha \leq 1$) and by $W_n = n^{1-\alpha}$ (for $0 < \alpha < 1$). Note that the second definition gives

$$w_n = n^{1-\alpha} - (n-1)^{1-\alpha} = \int_{n-1}^{n} \frac{1-\alpha}{t^\alpha}\,dt,$$

and hence

$$\frac{1-\alpha}{n^\alpha} \leq w_n \leq \frac{1-\alpha}{(n-1)^\alpha}.$$

Several of our estimations will be expressed in terms of the zeta function. It will be helpful to recall that $\zeta(1+\alpha) = 1/\alpha + r(\alpha)$ for $\alpha > 0$, where $1/2 \leq r(\alpha) \leq 1$ and $r(\alpha) \to \gamma$ (Euler's constant) as $\alpha \to 0$.

Example. Let A be the averaging operator, defined by $y = Ax$, where $y_n = \frac{1}{n}(x_1 + \cdots + x_n)$. By Hardy's inequality, $\|A\|_p = p^*$ $(= p/(p-1))$. Let Then $\|x_n\|_p^p = n$, while

$$\begin{aligned}\|A(x_n)\|_p^p &= n + n^p\sum_{i>n}\frac{1}{i^p}\\ &\leq n + n^p\frac{1}{(p-1)n^{p-1}}\\ &= n\left(1+\frac{1}{p-1}\right)\\ &= np^*,\end{aligned}$$

(The above inequality holds by integral estimations.)

3. The Hilbert operator

We consider the Hilbert operator H, with matrix $h_{i,j} = 1/(i+j)$. This satisfies conditions (i),(ii) and (iii) of Lemma 2.1. Hilbert's classical inequality states that $\|H\|_p = \pi/\sin(\pi/p)$ for $p > 1$. For the Lorentz space $d(w,1)$, with either of our choices of w, it was shown in [Lash 2] that $\|H\|_{w,1} = \pi/\sin\alpha\pi$. The analogous operator in the continuous case is defined by

$$(Hf) = \int_0^\infty \frac{f(y)}{x+y}dy.$$

By the Theorem 319 of [HLP], with

$$K(x,y) = \frac{x^{\alpha/p}}{y^{\alpha/p}(x+y)},$$

one can show that when $w(x) = 1/x^\alpha$, we have $\|H\|_{L_p(w)} \leq \pi/\sin[(1-\alpha)\pi/p]$. As we show below, this is the exact value, even when we restrict to decreasing functions. Numerous studies have been made of more general weighting functions in the continuous case (e.g. [AH], [AM]). For the discrete case, we will show that the above value is correct with either of our choices of w. Our method is a recognizable adaptation of that of [HLP], though some exact care is needed to ensure that it applies to the case $W_n = n^{1-\alpha}$. Let $0 < \alpha < 1$. As with most studies of the Hilbert operator, we use the well-known integral

$$\int_0^\infty \frac{1}{t^\alpha(t+c)}\,dt = \frac{\pi}{c^\alpha \sin\alpha\pi}.$$

Write

$$y_n = \int_{n-1}^{n}\frac{1}{t^\alpha}\,dt.$$

Note that $y_n \geq 1/n^\alpha$.

Lemma 3.1. With this notation, we have for each $j \geq 1$,

$$\sum_{i=1}^{\infty}\frac{1}{i^\alpha(i+j)} \leq \sum_{i=1}^{\infty}\frac{y_i}{i+j} \leq \frac{\pi}{j^\alpha\sin(\alpha\pi)}.$$

Proof. Clearly,

$$\frac{y_i}{i+j} = \frac{1}{i+j}\int_{i-1}^{i}\frac{1}{t^\alpha}\,dt \leq \int_{i-1}^{i}\frac{1}{t^\alpha(t+j)}\,dt.$$

The statement follows, by the integral quoted above. Now let $0 < \alpha < 1$. Write $u_n = y_n$ and $v_n = (1-\alpha)u_n$. In our previous terminology, v is our "second choice of w". Clearly, an operator will have the same norm on $d(v,p)$ and on $d(u,p)$. Note that by Hölder's inequality for integrals, $u_n^r \leq y_n$ for $r > 1$. **Theorem 3.2**. Let

H be the operator with matrix $h_{i,j} = 1/(i+j)$, and let $p > 1$. Let $0 < \alpha < 1$, and let $w_n = 1/n^\alpha$ and $u_n = \int_{n-1}^n t^{-\alpha} dt (or\ W_n = n^{(1-\alpha)}$. Then

$$\|H\|_{w,p} = \|H\|_{u,p} = \frac{\pi}{\sin[(1-\alpha)\pi/p]}.$$

Also, if $w_n = n^\alpha$, where $0 < \alpha < p-1$, then

$$\|H\|_{w,p} = \frac{\pi}{\sin[(1+\alpha)\pi/p}.$$

Proof. (i) Write $M = \pi/\sin[(1-\alpha)\pi/p]$. Let $w_n = n^{-\alpha}$, where $1-p < \alpha < 1$. Now $\|H\|_{w,p} = \|B\|_p$, where B has matrix $b_{i,j} = (j/i)^{\alpha/p}/(i+j)$. In Lemma 2.2, take $s_i = t_i = i$, and let K_1, K_2 be defined as before. Then

$$b_{i,j} s_i^{1/p} t_j^{-1/p} = \frac{1}{i+j}\left(\frac{i}{j}\right)^{(1-\alpha)/p}.$$

By Lemma 3.1, it follows that $K_1 \le M$. Similarly, $K_2 \le M$. Now let $0 < \alpha < 1$, and let u, w be as stated. We show in fact that $\|H\|_{w,u,p} \le M$. It then follows that $\|H\|_{u,p} \le M$, since $\|x\|_{w,p} \le \|x\|_{u,p}$ for all x. Note that $\|H\|_{w,u,p} = \|B\|_p$, where now

$$b_{i,j} = \frac{1}{i+j}(j^\alpha u_i)^{1/p}.$$

Take $s_i = u_i^{-1/\alpha}$ and $t_j = j$. Then

$$b_{i,j}(s_i/t_j)^{1/p} = \frac{1}{i+j}(j^\alpha u_i)^{(\alpha-1)/\alpha p}.$$

By Lemma 3.1,

$$\sum_{j=1}^\infty \frac{j^{(\alpha-1)/p}}{i+j} \le i^{(\alpha-1)/p} M.$$

Since $i^{-1} \le u_i^{1/\alpha}$, we have $i^{(\alpha-1)/p} \le u_i^{(1-\alpha)/\alpha p}$, and hence $K_1 \le M$. Also,

$$b_{i,j}(t_j/s_i)^{1/p^*} = \frac{1}{i+j}(j^\alpha u_i)^t,$$

where

$$t = \frac{1}{p} + \frac{1}{\alpha p^*}.$$

Note that $t > 1$, so as remarked above, we have $u_i^t \le y_i$. By Lemma 3.1 again,

$$\sum_{i=1}^\infty \frac{y_i}{i+j} \le \frac{M'}{j^{\alpha t}},$$

where $M' = \pi/\sin \alpha t \pi$. Now

$$\alpha t = \frac{\alpha}{p} + \frac{1}{p^*} = 1 - \frac{1-\alpha}{p},$$

so that $M' = M$, and hence $K_2 \le M$. The statement follows.

(ii) To show that equality holds, we must show that there is a *decreasing* sequence x such that $\|Hx\|_{w,p}/\|x\|_{w,p}$ is arbitrarily close to M (and similarly with u replacing w). For the moment, consider w. Take $r = (1-\alpha)/p$, so that $\alpha + rp = 1$. Fix N, and let

$$x_j = \begin{cases} 1/j^r & \text{for } j \le N, \\ 0 & \text{for } j > n. \end{cases}$$

Then $\sum_{j=1}^\infty w_j x_j^p = \sum_{j=1}^N \frac{1}{j}$. Let $y = H(x)$. Then for all i,

$$y_i = \sum_{j=1}^N \frac{1}{j^r(i+j)} \ge \int_1^N \frac{1}{t^r(t+i)}\, dt.$$

Now

$$\int_0^\infty \frac{1}{t^r(t+i)}\, dt = \frac{K}{i^r}$$

(with K as before). Also,

$$\int_0^1 \frac{1}{t^r(t+i)}\, dt \le \frac{1}{i}\int_0^1 \frac{1}{t^r}\, dt = \frac{1}{i(1-r)}$$

and

$$\int_N^\infty \frac{1}{t^r(t+i)}\, dt \le \int_N^\infty \frac{1}{t^{r+1}}\, dt = \frac{1}{rN^r},$$

so (again using the integral quoted above)

$$y_i \ge \frac{M}{i^r} - \frac{1}{i(1-r)} - \frac{1}{rN^r}.$$

By the elementary inequality $(a-x)^p \ge a^p - pa^{p-1}x$, we have

$$y_i^p \ge \frac{M^p}{i^{rp}} - \frac{pM^{p-1}}{i^{r(p-1)}}\left(\frac{1}{i(1-r)} + \frac{1}{rN^r}\right).$$

Since $w_i = 1/i^\alpha$ and $\alpha + rp = 1$, we obtain

$$w_i y_i^p \ge \frac{M^p}{i} - pM^{p-1}\left(\frac{1}{(1-r)i^{2-r}} + \frac{1}{rN^r i^{1-r}}\right).$$

Now

$$\sum_{i=1}^N \frac{1}{i^{1-r}} \le \int_0^N \frac{1}{t^{1-r}}\, dt = \frac{N^r}{r}.$$

Hence

$$\sum_{i=1}^{\infty} w_i y_i^p \geq \sum_{i=1}^{N} w_i y_i^p \geq M^p \sum_{i=1}^{N} \frac{1}{i} - pM^{p-1} c_r,$$

where

$$c_r = \frac{\zeta(2-r)}{1-r} + \frac{1}{r^2}.$$

This proves the statement for w. Since $w_n \leq u_n \leq c w_n$ for a constant c, a slight modification of the above reasoning gives the same conclusion for u. □

Note. In the continuous case, the second half of the proof must be modified slightly to ensure convergence of the integral at 0. This is best achieved by taking $\alpha + rp = 1 + \varepsilon$ and defining $f(x)$ to be $1/x^r$ for $x \geq 1$ and 1 for $0 < x < 1$.

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