Special matrices for descriptor systems with restrictions *

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Abstract:- A special structure of a kind of matrices, $r-$block monomial and $r-$block diagonal matrices, is introduced. Theirs nonnegative Drazin inverses have been studied. The influence of these matrices on the positiveness of the trajectory of a descriptor system has been analyzed.

Key–Words:- Drazin inverse, nonnegative matrix, block monomial matrix, descriptor system.

1 Introduction

In control theory the structure of coefficients matrices plays an important role because structural properties and the solution of the discrete-time control system are given using these matrices.

When the interest of the problem is based on some restrictions over the behavior of the system, then the influence of the coefficient matrices is increased.

We focus our attention on the case when nonnegative restrictions are considered. In this problem appears nonnegative matrices, that is matrices whose entries are nonnegative.

In this paper, a discrete-time descriptor system is considered, that is, a control system given by

$$Ex\,(k+1) \quad = \quad Ax\,(k) + Bu\,(k)\,.$$

When E is a singular matrix the solution of this discrete-time descriptor control system involves Drazin inverses of the matrices E and A. Now, it is worthwhile studying conditions on the nonnegativity of the Drazin inverse.

Not many results are known on the nonnegativity of generalized inverses. For instance, in [2] some properties on the nonnegativity of Moore-Penrose inverse are studied, in [8] a characterization of nonnegative group inverse of a nonnegative block lower triangular matrix is given, and in [1] the structure of a nonnegative regular matrix, that is a matrix such that it admits a nonnegative generalized inverse, is analyzed.

In this paper, we obtain some conditions on the coefficient matrices of a descriptor system in order to the trajectory of the system belongs nonnegative when a nonnegative admissible inital state and a nonnegative control sequence are considered. For that, the $r-$block diagonal matrix concept is introduced and some properties of this kind of matrices are studied. Then, these properties allow us to obtain the desired results.

2 Block monomial matrices

Consider a discrete-time linear descriptor system given by

$$Ex\,(k+1) \quad = \quad Ax\,(k) + Bu\,(k)\,, \qquad (1)$$

where $E,\ A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $k \in \mathbb{Z}_{+}$. The system is denoted by (E, A, B). It is well-known (see [6] and [7]) that if the pair (E, A) is regular, that is, there exists $\lambda \in \mathbb{C}$ such that $\det(\lambda E - A) \neq 0$, then the system has solution.

The aim of this paper is to study a special kind of matrices E, A and B for obtaining the positiveness

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of the trajectory of the system. We also analyze the nonnegativity of the Drazin inverse of these matrices because this solution is given in terms of Drazin inverse of the matrices E and A.

Firstly, we give some new definitions.

Definition 1 *A block permutation matrix P is a matrix decomposed in blocks such that it has precisely one identity block matrix in each row and column and the rest of blocks are zero.*

Definition 2 *Let A be an $n \times n$ nonnegative matrix. A is a $r-$block monomial matrix if there exists a block permutation matrix P such that $AP =$ diag $[A_i]_{i=1}^{r}$, where each block A_i has rank one. If $P = I$ it is said that A is a $r-$block diagonal matrix.*

In the following proposition we can see that a special structure of blocks A_i , $i = 1, \dots, r$, characterizes the symmetry of the matrix.

Lemma 1 *A nonnegative matrix A is symmetric matrix of rank one if and only if $A = vv^T$.*

Proof. If $A = vv^T$, A is a symmetric matrix of rank one.

On the other hand, if A is a symmetric matrix of rank one,

$$\begin{aligned} A &= \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix} = \\ &= a_{11} \begin{pmatrix} 1 & \alpha_1 & \cdots & \alpha_{n-1} \\ \alpha_1 & \alpha_1\alpha_1 & \cdots & \alpha_1\alpha_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n-1} & \alpha_{n-1}\alpha_1 & \cdots & \alpha_{n-1}\alpha_{n-1} \end{pmatrix} \\ &= a_{11} \begin{pmatrix} 1 \\ \alpha_1 \\ \vdots \\ \alpha_{n-1} \end{pmatrix} \begin{pmatrix} 1 & \alpha_1 & \cdots & \alpha_{n-1} \end{pmatrix}. \end{aligned}$$

Thus, $A = vv^T$. □

Without loss of generality we shall consider the $r-$block diagonal matrix $A =$ diag $[A_i]_{i=1}^{r}$ instead of $r-$block monomial matrix.

Next, we study the Drazin inverse of a $r-$block diagonal matrix, in order to obtain some nonnegative properties of these matrices, for that, we shall use the index of the matrix A. Remember that this index is the smallest nonnegative integer such that rank(A^q) =rank$\big(A^{q+1}\big)$. Along this paper it is denoted by ind(A).

Proposition 1 *Consider a $r-$block diagonal matrix A with $A_i = v_i v_i^T$, and $v_i \geq 0$, $i = 1, \dots, r$.*

(i) ind(A)=1.

(ii) $A^D =$ diag $\left[A_i^D\right]_{i=1}^{r}$, where

$$A_i^D = \frac{v_i v_i^t}{\|v_i\|_2^4},$$

for $i = 1, \dots, r$.

(iii) AA^D is a symmetric matrix of rank one.

Proof. (i) Since $A_i = v_i v_i^T$, then rank$(A_i) = 1$ and

$$A_i^2 = v_i^T v_i A_i.$$

Thus, rank(A_i^2) =rank(A_i) , and ind(A_i)=1. From the diagonal structure of matrix A, then

$$\text{rank}(A^2) = \text{rank}(A) = r$$

and ind(A)=1.

(ii) The Drazin inverse of A is a matrix A^D which satisfies the following conditions (see [3] and [5])

$$\begin{aligned} A^D A A^D &= A^D \\ A^D A &= A A^D \\ A^D A^{k+1} &= A^k, \ k \geq \text{ind}(A). \end{aligned} \tag{2}$$

Consider $A =$ diag $[A_i]_{i=1}^{r}$ such that $A_i = v_i v_i^T$, and its Drazin inverse $A^D =$ diag $\left[A_i^D\right]_{i=1}^{r}$. For $i = 1, \dots, r$, it is easy to prove that the matrix

$$\frac{v_i v_i^t}{\|v_i\|_2^4},$$

satisfies the Drazin inverse conditions (2).

(iii) From (ii)

$$A_i A_i^D = \frac{A_i}{\|v_i\|_2^2}.$$

Thus, AA^D is also a symmetric matrix. □

Note that from above proposition the Drazin invese of a r-block diagonal matrix is nonnegative.

3 Positive descriptor systems

Consider the discrete-time linear descriptor system (E, A, B) given in (1). For an initial state $x(0) \in \mathcal{X}_0$, where $\mathcal{X}_0$ denotes the admissible initial conditions set, and a sequence control $u(j)$, $j = 0, 1, \dots, k+q-1$, the state solution of this system is given by

$$x\left(k\right) =$$

$$\begin{aligned} &= \left(\widehat{E}^D\widehat{A}\right)^k \widehat{E}^D\widehat{E}x\left(0\right) \qquad (3)\\ &\quad + \sum_{i=0}^{k-1} \widehat{E}^D \left(\widehat{E}^D\widehat{A}\right)^{k-i-1} \widehat{B}u\left(i\right)\\ &\quad - \left(I - \widehat{E}^D\widehat{E}\right) \sum_{i=0}^{q-1} \left(\widehat{E}\widehat{A}^D\right)^i \widehat{A}^D\widehat{B}u\left(k+i\right), \end{aligned}$$

where

$$\begin{aligned} \widehat{E} &= (\lambda E - A)^{-1} E,\\ \widehat{A} &= (\lambda E - A)^{-1} A,\\ \widehat{B} &= (\lambda E - A)^{-1} B, \end{aligned}$$

and q=ind$(\widehat{E})$. Note that, $\widehat{E}\widehat{A} = \widehat{A}\widehat{E}$. This property is a basic condition to obtain the solution of the system in terms of the Drazin inverses of matrices $\widehat{E}$ and $\widehat{A}$.

We give the following definition used along the paper.

Definition 3 *(see [4]) The system (1) is positive if, for every $x(0) \in \mathcal{X}_0 \cap \mathbb{R}^n_+$, and for every nonnegative control sequence $u(j) \geq 0$, $j \geq 0$, the state trajectory belong to $\mathbb{R}^n_+$, that is*

$$x\left(k\right) = x\left(k, x_0, u\left(\cdot\right)\right) \in \mathbb{R}^n_+, \forall k \geq 0.$$

Firstly, we give the following proposition for autonomous case. .

Proposition 2 *Consider a discrete-time descriptor system (E, A). If E is a $r-$block diagonal symmetric matrix, A is a nonnegative matrix and the matrices E and A commute, then the system is positive, for every initial state $x(0) \in \mathcal{X}_0 \cap \mathbb{R}^n_+$.*

Proof. By proposition 1, E^D is a nonnegative matrix and then

$$x\left(k\right) = \left(E^D A\right)^k E^D E x\left(0\right) \geq 0,$$

for every initial state $x(0) \in \mathcal{X}_0 \cap \mathbb{R}^n_+$. $\square$

From the solution (3) it is worthwhile studying the matrix $(I - E^D E)$ when the case with controls is considered.

We denote

$$\mathcal{C}_v = \left\{x \in \mathbb{R}^n : \ v^T x \leq \min(x_i, \ i = 1 \dots, n)\right\}.$$

Proposition 3 *Consider the matrix $I - E^D E$, where E is a $r-$block diagonal symmetric matrix. Then,*

(i) rank($I - E^D E) = n - r$.

(ii) The vector $\left(I - E^D E\right) x$ is nonnegative, for all $x \in \mathcal{C}_v$.

Proof. (i) The matrices $E^D E$ and $I - E^D E$ are projectors matrices, because they verify the projector property $P^2 = P$. Thus, $E^D E$ and $I - E^D E$ are complementary matrices and

$$\text{rank } \left(I - EE^D\right) = n - \text{rank } \left(EE^D\right) = n - r.$$

(ii) For simplify the proof we consider that matrix E has only one block, $E = vv^T$ with

$$v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}.$$

For the construction of $E^D E$, the matrix $I - E^D E$ is given by

$$I - E^D E =$$

$$= \frac{1}{\|v\|_2^2} \begin{pmatrix} v^T v - v_1^2 & -v_1 v_2 & \dots & -v_1 v_n \\ -v_1 v_2 & v^T v - v_2^2 & \dots & -v_2 v_n \\ \vdots & \vdots & \ddots & \vdots \\ -v_1 v_n & -v_2 v_n & \dots & v^T v - v_n^2 \end{pmatrix}.$$

That is, this is a symmetric matrix. We consider

$$\left(I - E^D E\right) x = b.$$

By a technical process can be proved that b cannot be negative, and then its entries must be nonnegative. $\square$

In the next result a characterization of positive descriptor system is given.

Proposition 4 *Consider a discrete-time descriptor system* (E, A, B) *with* $EA = AE$ *and* E *is a* $r-$*block diagonal symmetric matrix,* $A \geq 0$, $B \geq 0$. *Then,* $\left(I - E^D E\right) A^D B \leq 0$ *if and only if the system is positive.*

Proof. Since $q = 1$ and writting

$$x(k) = E^D E x(k) + \left(I - E^D E\right) x(k),$$

the trajectory of discrete-time descriptor system (E, A, B) is given by

$$\begin{aligned} E^D E x(k) &= \left(E^D A\right)^k E^D E x(0) \\ &\quad + \sum_{i=0}^{k-1} E^D \left(E^D A\right)^{k-i-1} B u(i) \\ \left(I - E^D E\right) x(k) &= -\left(I - E^D E\right) A^D B u(k). \end{aligned}$$

Using the hypothesis and by proposition 1, $E^D \geq 0$ and $E^D E x(k)$ is nonnegative for every $x(0) \in \mathcal{X}_0 \cap \mathbb{R}^n_+$, and for every nonnegative control sequence $u(k) \geq 0$, $k \geq 0$.

If $\left(I - E^D E\right) A^D B \leq 0$, then $x(k)$ is nonnegative. If the solution of the system is nonnegative, when consider $x(0) = 0$,

$$\left(I - E^D E\right) A^D B u(k) \leq \sum_{i=0}^{k-1} E^D \left(E^D A\right)^{k-i-1} B u(i),$$

for every nonnegative control sequence $u(k) \geq 0$, $k \geq 0$. In particular, taking $k = 1$,

$$\left(I - E^D E\right) A^D B u(1) \leq E^D B u(0).$$

Considering $u(0) = 0$ and $u(1) = e_i$, then $\left(I - E^D E\right) A^D B \leq 0$. $\square$

Note that if $E = \operatorname{diag}\left[v_i v_i^T\right]_{i=1}^r$ with $\|v_i\|_2 = 1$, $i = 1, \ldots, r$, then, under the hypothesis of the above proposition, the system is positive if and only if $A^D B \leq E A^D B$.

Now, we discuss the positiveness of the system using proportional state-feedbacks. Note that, if we apply a state feedback $u(k) = Fx(k)$ to the system (E, A, B) given in (1), then the closed-loop system is given by

$$Ex(k+1) = (A + BF) x(k)$$

and it is easy to prove the following results.

Corollary 1 *Consider a discrete-time descriptor system* (E, A, B). *Suppose that* E *is a* $r-$*block diagonal symmetric matrix and matrices* E *and* A *commute. If* $A \geq 0$, *there exists a state-feedback* $u(k) = Fx(k)$ *such that* $BF \geq 0$ *and*

$$\left(I - E^D E\right) A^D B F = 0$$

then the system is positive.

Proposition 5 *Consider a discrete-time descriptor system* (E, A, B). *Suppose that* E *is a* $r-$*block diagonal symmetric matrix. If there exists a state-feedback* $u(k) = Fx(k)$ *such that*

$$A + BF \geq 0$$

with $EBF = BFE$, *then the closed–loop system is positive.*

4 Conclusion

In this work, the concepts of $r-$block monomial and diagonal matrices are introduced. Properties of these matrices have been studied. In particular, the nonnegativity of the Drazin inverse of a $r-$block diagonal symmetric matrix is shown. Some properties of this kind of matrices are used to obtain a nonnegative trajectory of a discrete-time descriptor system. Finally, the positiveness of the system using proportional state-feedbacks is discussed.

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