Extension of Hajek-Renyi inequality for negative dependence random variables

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Abstract: *In this paper we extended the Hajek-Renyi inequality, and obtain strong laws of large numbers for negative dependence (ND) random variables by use in this inequality and martingale techniques.*

Key Words: *Hajek-Renyi inequality Negative Dependent, Strong Law of Large Numbers, martingale.*

1.Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of random variable's defined on a probability space $(\Omega,\ \mathcal{F},\ p)$. Ebrahimi and Ghosh [4] introduced the following definition.

Definition 1.The random variables $X_1, \cdots, X_n$ are said to be ND if we have

$$P[\bigcap_{j=1}^{n}(X_j \leq x_j)] \leq \prod_{j=1}^{n} P[X_j \leq x_j], \quad (1)$$

and

$$P[\bigcap_{j=1}^{n}(X_j > x_j)] \leq \prod_{j=1}^{n} P[X_j > x_j], \quad (2)$$

for all $x_1, \cdots, x_n \in R$. An infinite sequence $\{X_n,\ n \geq 1\}$ is said to be ND if every finite subset $X_1, \cdots, X_n$ is ND. The conditions (1) and (2) are equivalent for $n = 2$, but these do not agree for $n \geq 3$ (see Bozorgnia,et. all [3])

Let $S_n = \sum_{i=1}^{n} X_i$. The problem convergence with probability one of $\frac{S_n - ES_n}{n}$ for sequence of negative dependence random variable's was studied by Bozorgnia et.all [3]. In addition strong laws of large numbers and complete convergence for ND random variable's were studied by Amini and Bozorgnia [1],[2]. We will proved strong laws of large numbers for ND random variables by use Hajek-Renyi inequality. The Hajek-Renyi (1955) prove the following important inequality. If $\{X_n, n \geq 1\}$ is a sequence of independent random variables with $E(X_n) = 0$, $E(X_n^2) < \infty$, $n \geq 1$ and $\{b_n,\ n \geq 1\}$ is a positive nondecreasing sequence of real numbers. Then for every $\varepsilon > 0$, and positive $m < n$,

$$P[\max_{m \leq j \leq n} \frac{|S_j|}{b_j} \geq \varepsilon] \leq \frac{1}{\varepsilon^2}(\sum_{j=m+1}^{n} \frac{\sigma_j^2}{b_j^2} + \sum_{j=1}^{m} \frac{\sigma_j^2}{b_m^2}), \quad (3)$$

where $\sigma_j^2 = Var(X_j)$.

This inequality has been studied by many outhors, the latest literature is given by J.Liu, S.Gan and P.Chen [7] for negative association random variables. In this paper,we extend this inequality for ND random variables by martingale technique and using it we proved some strong limit theorems . We need to following lemmas for next

sections.

Lemma 1(Kronecker) If $\{a_n\}$, $\{b_n\}$ are sequences of real numbers with,

$b_n \uparrow \infty$, if $\sum_{j=1}^{\infty} \frac{a_j}{b_j}$ converging, then

$$\frac{1}{b_n} \sum_{j=1}^{n} a_j \longrightarrow 0.$$

Lemma 2([5]) The sequence $\{X_n,\ n \geq 1\}$ converges to X with probability one if and only if for every $\varepsilon > 0$,

$$\lim_{n \to \infty} P[\sup_{k \geq n} |X_k - X| > \varepsilon] = 0.$$

2. Extension of Hajek-Renyi inequality

In this section, we extend Hajek-Renyi inequality for ND random variables, where $E[X_n|\mathcal{F}_{n-1}] = 0,\ \ \mathcal{F}_n = \sigma(X_1, \cdots, X_n)$ for every $n \geq 1$.

Lemma 3([6]) Let $\{X_n, \mathcal{F}_n,\ n \geq 1\}$ be a martingale and $\{b_n,\ n \geq 1\}$ be a sequence of positive nondecreasing real numbers,then for every $\varepsilon > 0$,

$$P[\max_{1 \leq k \leq n} \frac{|X_k|}{b_k} > \varepsilon] \leq$$

$$\frac{1}{\varepsilon^1}(b_1^{-1} EX_1^+ + \sum_{k=2}^{n} b_k^{-1}(EX_k^+ - EX_{k-1}^+)), \qquad (4)$$

Remark Under the assumption of lemma 3 for every $1 \leq m \leq n$, we have,

$$P[\max_{m \leq k \leq n} \frac{|X_k|}{b_k} > \varepsilon] = P[\max_{1 \leq j \leq n-m+1} \frac{|X_{j+m-1}|}{b_{j+m-1}} > \varepsilon]$$

$$\leq \frac{1}{\varepsilon^1}(b_m^{-1} EX_m^+ + \sum_{j=2}^{n-m+1} b_{j+m-1}^{-1}(EX_{j+m-1}^+ - EX_{j+m-2}^+))$$

$$= \frac{1}{\varepsilon^1}(b_m^{-1} EX_m^+ + \sum_{k=m+1}^{n} b_k^{-1}(EX_k^+ - EX_{k-1}^+))$$

Theorem 1. Let $\{X_n,\ n \geq 1\}$ be a sequence of ND random variables with $E[X_n|\mathcal{F}_{n-1}] = 0$ and $EX_n^2 < \infty, n \geq 1$ and $\{b_n,\ n \geq 1\}$ be a sequence of positive nondecreasing real numbers, then for every $\varepsilon > 0$,

i)

$$P[\max_{1 \leq k \leq n} \frac{|S_k|}{b_k} \geq \varepsilon] \leq \frac{1}{\varepsilon^2} \sum_{j=1}^{n} \frac{\sigma_j^2}{b_j^2}$$

ii)

$$P[\max_{m \leq k \leq n} \frac{|S_k|}{b_k} \geq \varepsilon] \leq \frac{1}{\varepsilon^2}(\sum_{k=m+1}^{n} \frac{\sigma_k^2}{b_k^2} + \sum_{k=1}^{m} \frac{\sigma_k^2}{b_m^2})$$

Proof. Since $E[X_n|\mathcal{F}_{n-1}] = 0$, then $\{S_n,\ \mathcal{F}_n,\ n \geq 1\}$ is a martingale and hence $\{|S_n|,\ \mathcal{F}_n,\ n \geq 1\}$ is a submartingale. In addition if h is an increasing function then $\{h(|S_n|),\ \mathcal{F}_n,\ n \geq 1\}$ is a submartingale, thus by lemma 3 for every $\varepsilon > 0$, we have

i)

$$P[\max_{1 \leq k \leq n} \frac{|S_k|}{b_k} \geq \varepsilon] \leq P[\max_{1 \leq k \leq n} \frac{S_k^2}{b_k^2} \geq \varepsilon^2]$$

$$\leq \varepsilon^{-2}[b_1^{-2} EX_1^2 + \sum_{k=2}^{n} b_k^{-2}(ES_k^2 - ES_{k-1}^2)] \leq$$

$$\frac{1}{\varepsilon^2} \sum_{j=1}^{n} \frac{\sigma_j^2}{b_j^2}$$

ii) By above remark for $1 \leq m < n$, we obtain

$$P[\max_{m \leq k \leq n} \frac{|S_k|}{b_k} \geq \varepsilon] \leq P[\max_{m \leq k \leq n} \frac{S_k^2}{b_k^2} \geq \varepsilon^2]$$

$$\leq \frac{1}{\varepsilon^2}[b_m^{-2} ES_m^2 + \sum_{k=m+1}^{n} b_k^{-2}(ES_k^2 - ES_{k-1}^2)] \leq$$

$$\frac{1}{\varepsilon^2}(\sum_{k=1}^{m} \frac{\sigma_k^2}{b_m^2} + \sum_{k=m+1}^{n} \frac{\sigma_k^2}{b_k^2}).$$

Hence complete the proof. $\square$.

Corollary 1. Under the assumption of theorem 1 we have

i)

$$P[\max_{1 \leq k \leq n} \frac{|S_k|}{b_n} \geq \varepsilon] \leq \frac{1}{\varepsilon^2} \sum_{j=1}^{n} \frac{\sigma_j^2}{b_j^2}$$

ii)

$$P[\max_{m \leq k \leq n} \frac{|S_k|}{b_n} \geq \varepsilon] \leq \frac{1}{\varepsilon^2}(\sum_{k=m+1}^{n} \frac{\sigma_k^2}{b_k^2} + \sum_{k=1}^{m} \frac{\sigma_k^2}{b_m^2})$$

3. Some Strong Convergence Theorems

In this section, by using Theorem 1 we can prove the following theorem for negative dependence random variable's .

Theorem 2. Let $\{X_n,\ n \geq 1\}$ be a sequence of ND random variables with $E[X_n|\mathcal{F}_{n-1}] = 0,\ \ n \geq 1$ and $\{b_n,\ n \geq 1\}$ be a sequence of positive nondecreasing real numbers. If $\sum_{n=1}^{\infty} \frac{\sigma_n^2}{b_n^2} < \infty$, then for every $0 < \beta < 2$

i) $E(\sup_n (\frac{|S_n|}{b_n})^\beta) < \infty$.

ii) If $b_n \longrightarrow \infty$, as $n \to \infty$, then

$$\frac{S_n}{b_n} \longrightarrow 0, \quad W.P.1.$$

Proof.

i) We have

$$E(\sup_n (\frac{|S_n|}{b_n})^\beta) < \infty \quad \Leftrightarrow$$

$$\int_1^\infty P[\sup_n \frac{|S_n|}{b_n} > t^{\frac{1}{\beta}}]dt < \infty.$$

Now by lemma 2 and Hajek-Renyi inequality we obtain

$$P[\sup_n \frac{|S_n|}{b_n} > t^{\frac{1}{\beta}}] = \lim_{m\to\infty} P[\max_{1\le n\le m} \frac{|S_n|}{b_n} > t^{\frac{1}{\beta}}]$$

$$\le \frac{1}{t^{\frac{2}{\beta}}} \sum_{n=1}^\infty \frac{\sigma_n^2}{b_n^2}.$$

Hence

$$\int_1^\infty P[\sup_n \frac{|S_n|}{b_n} > t^{\frac{1}{\beta}}]dt \le$$

$$\int_1^\infty t^{\frac{-2}{\beta}} \sum_{n=1}^\infty \frac{\sigma_n^2}{b_n^2} dt = O(1) \sum_{n=1}^\infty \frac{\sigma_n^2}{b_n^2} < \infty,$$

this complete the proof.

ii) By Theorem 1, for every $\varepsilon > 0$,

$$P[\sup_{m\le k} \frac{|S_k|}{b_k} > \varepsilon] = \lim_{n\to\infty} P[\max_{m\le k\le n} \frac{|S_k|}{b_k} > \varepsilon]$$

$$\frac{1}{\varepsilon^2}(\sum_{j=1}^m \frac{\sigma_j^2}{b_m^2} + \sum_{j=m+1}^\infty \frac{\sigma_j^2}{b_j^2}),$$

now by the assumption of $\sum_{j=1}^\infty \frac{\sigma_j^2}{b_j^2} < \infty$, and Kronecker's lemma we obtain

$$\lim_{m\to\infty} P[\sup_{k\ge m} \frac{|S_k|}{b_k} > \varepsilon] = 0,$$

hence lemma 2 complete the proof. □

Corollary 2. Under the assumption of theorem 2, if $\sup_n \sigma_n^2 < \infty$, then for every $\alpha > \frac{1}{2}$.

i)

$$P[\sup_{j\ge m} \frac{|S_j|}{j^\alpha} > \varepsilon] \le$$

$$\frac{1}{\varepsilon^2}(\sum_{j=1}^m \frac{1}{m^{2\alpha}} + \sum_{j=m+1}^\infty \frac{1}{j^{2\alpha}}) \sup_n \sigma_n^2.$$

ii) $\frac{S_n}{n^\alpha} \longrightarrow 0 \quad W.P.1.$

iii) For every $0 < \beta < 2$,

$$E(\sup_n (\frac{|S_n|}{n^\alpha})^\beta) < \infty.$$

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