DKA method for single variable holomorphic functions

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Abstract: - Durand-Kerner-Aberth (DKA) method for single variable holomorphic functions are treated. Finite series approximation of the single variable holomorphic function is used to find local roots of single variable holomorphic functions instead of algebraic equation. We can find roots of single variable holomorphic functions in wide region by continuing local solutions.

Key-Words: - Durand-Kerner-Aberth (DKA) method, root finding

1 Introduction

Advantage of Newton's iteration method for non-linear equation or algebraic equation is well-known and detailed explanations and its applications are found in many publications [1]. However, bridge for the gap between algebraic equation and analytic function with this method was not giving much. To cover this gap [2] is worked towards constructing mathematical base. Here we will attempt to cover this gap in actual problem. In this article, root finding method DKA [3, 4] and 3rd order methods are treated from both algebraic and analytic points. From this point, DKA method resembles to Euclidian elimination (algebra) and Weierstrass preparation theorem (analysis). Numerical algorithms are not well investigated from above mixed point up to this point. We expect to obtain new insight from this point.

1.1 DKA method and 3rd order method

Durand-Kerner-Aberth method is numerical root finding algorithm for algebraic equation. Consider n-th degree algebraic equation with real number coefficient,

\[
P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0 = 0, \quad a_i \neq 0, \quad z \in \mathbb{C}.
\]

The n number of roots can be found numerically by following Newton's method,

\[
\begin{align*}
\dot{z}^{(k)} &= \left[ \begin{array}{c}
z_1^{(k)} \\
\vdots \\
z_n^{(k)}
\end{array} \right], \\
f(z) &= \left[ \begin{array}{c}
f_1(z) \\
\vdots \\
f_n(z)
\end{array} \right], \\
J(z) &= \left( \frac{\partial f_j(z)}{\partial z_i} \right).
\end{align*}
\]

(3) \( z^{(k+1)} = z^{(k)} - J(z^{(k)})^{-1} f(z^{(k)}) \),

\( k \) : iteration number and

\[
\begin{align*}
\phi_m(z_1^{(k)}, z_2^{(k)}, \ldots, z_n^{(k)}) &= \sum_{i_1 < i_2 \cdots < i_m} z_{i_1}^{(k)} \cdots z_{i_m}^{(k)}, \quad 1 \leq m, \\
f_i(z_1, z_2, \ldots, z_n) &= (-1)^i \phi_i(z_1, z_2, \ldots, z_n) - a_i = 0, \quad i = 1, 2, \ldots, n.
\end{align*}
\]

Then (3) can be written as

\[
(4) \quad z^{(k+1)} = z^{(k)} - P(z^{(k)}) \prod_{i=1}^{n} (z^{(k)} - z^{(k)})^i, \quad i = 1, 2, \ldots, n.
\]

We can assume that (4) is holomorphic mapping, if \( z^{(1)} - z^{(i)} \neq 0 \) at every iteration step with proper initial conditions. It is an acceptable assumption, because initial conditions for \( z^{(1)} \) are given on the circle usually. Therefore (4) maps initial circle where initial values are given to the closed curve on that exact roots. It is known that such unique holomorphic function that maps circle to closed curve in \( \mathbb{C} \) exists. From algebraic geometrical point, (4) gives \( n \) numbers of generators for ideals, because each equation \( f_i(z) \) in (4) is independent. We can introduce open covering which gives restriction mapping, and etc., \( D_1^{(2)} \subseteq D_1^{(3)}, D_1^{(2)} \cap D_1^{(3)} \neq \emptyset \), \( D_2^{(2)} \cap D_3^{(2)} = \emptyset \), \( \rho_2^{(4)} : D_2^{(4)} \rightarrow D_2^{(4)}, \phi_2^{(4)} : z^{(4)} \rightarrow z^{(4)} \), see Fig.1.

![Fig.1 DKA method and region of the mapping.](image-url)
This property may give superior convergent property.

DKA method (4) can be deformed as,

\[ P(z^{(k)}_i) = (z^{(k)}_i - z^{(k-1)}_i) \prod_{j=1, j\neq i}^n (z^{(k)}_j - z^{(k-1)}_j), i = 1, 2, ..., n. \]

It resembles to following Euclidian elimination,

\[ P(z^{(k)}_i) = (z^{(k)}_i - z^{(k-1)}_i) h(z^{(k)}_i) + r(z^{(k)}_i), i = 1, 2, ..., n, \]

here \( h(z^{(k)}_i) = \prod_{j=1, j\neq i}^n (z^{(k)}_j - z^{(k-1)}_j) \). In addition

\[ r_i(z^{(k)}_i) = 0, i = 1, ..., n, \] then (6) becomes DKA method. DKA method seems numerical approximation of Euclidian elimination. Since it is known that polynomial functions in \( C \) are UFD, existence of numerical solution of the method is guaranteed. The following method is modification of DKA method, and it is called 3rd order method,

\[ z^{(k+1)} = z^{(k)} - P(z^{(k)}) / P(z^{(k)}) - P(z^{(k)} \sum_{j=1}^n \frac{1}{z^{(k)}_j - z^{(k-1)}_j}), i = 1, 2, ..., n. \]

2 DKA method for single variable holomorphic function

We can modify DKA method for holomorphic function \( f(w_i, z^{(k)}_i) \) with several variables,

\[ z^{(k+1)}_i = z^{(k)}_i - P(w_i, z^{(k)}_i) f(w_i, z^{(k)}_i) / \prod_{j=1, j\neq i}^n (z^{(k)}_j - z^{(k-1)}_j), i = 1, 2, ..., n. \]

\( P(w_i, z^{(k)}_i) \) is Weierstrass polynomial by \( z^{(k)}_i \), and \( u_i(w_i, z^{(k)}_i) \) are units. Existence of such division \( P=uf \) is guaranteed by Weierstrass preparation theorem [5]. Figure 2 illustrates image of unit \( U(z) \).

Holomorphic functions converge in the convergence domain of radius \( r \), (uniformly and absolutely converge with appropriate radius \( s < r \)), see Appendix also.

\[ f(z) = \sum_{n=0} a_n z^n, \ a_n \in C, \]

here Cauchy-Hadamard theorem for \( r \),

\[ \frac{1}{r} = \lim_{n \to \infty} \sup |a_n|, \ r(n) = (\sqrt[n]{|a_n|})^{1/n}. \]

We introduce variable transform, and approximate \( f(z) \) by finite series,

\[ z \to r(n)w, \ z = r(n)w, \ n: \text{int} < \infty, \]

\[ f(z) = f(r(n)w) = \sum_{n=0} a_n r'(n)w, \]

\[ = \sum_{n} b_n w, \ b_n = a_n r(n). \]

Then \( r \) of \( f(w) \) respect to coefficients \( b_i \) and independent variable \( w \) roughly equals to 1, and \( r(n) \) is polyradius that depends on \( f \) and \( n \). For example, approximation by finite series expansion of Bessel function, in this case \( r(20)=0.05746 \), behaves as Fig.3.

We will use polynomial function by finite series \( P(z) = \tilde{f}(z) \) instead of \( f(z) \) with DKA and 3rd order method to resolve local roots of \( f(z) \).

2.1 Uniform property of holomorphic (analytic) function

Firstly, we should confirm uniform property of the approximation functions by finite series (polynomials). Fortunately, uniform property of the approximation function is preserved in the appropriate treatment of the radius of the polydisc of original holomorphic function. If we use \( r(n) \) which is sufficiently smaller than radius of convergence, approximation function closes original function, because both functions are holomorphic functions which uniformly converge in the polydisc of \( r(n) \). Figure 4 shows behavior of \( r(n) \) respect to \( n \) for \( \exp(x) \). The figure shows that convergence radius \( r(n) \) of the approximation function becomes large by
increasing \( n \). Since \( a_n r(n)^n = O(1) \) as to approximation function, therefore polynomial function for approximation diverges where \( z > r(n) \). In general we need careful treatment for \( r(n) \) respect to the meaning of \( \lim \sup_{n \to \infty} \) in (11).

2.2 Convergence radius and its behavior by finite series approximation

With approximate radius \( r(n) \), we introduce variable transform \( z = r(n) w \). Discarding the term of \( z^{n+1} \) and higher in the series expansion of \( f(z) \), the approximation function is,

\[
(13) \quad f(w) = \tilde{f}(w) = \sum_{i=0}^{n} a_i r^n \theta w^i,
\]
then, Truncation error is \( O(a_n r^{n+1}(n)) \).

Since the convergence radius \( r \) by new coefficients \( b_i = a_i r(n) \) roughly equals to one, we only take up resolved roots by the DKA which are inside unit circle with following error correction. We define,

\[
\varepsilon = |\tilde{f}(z_j^{(n)}) - \delta|, \quad \delta = |f(z_j^{(n)})|, \quad \gamma = |f(z_j^{(n)}) - \tilde{f}(z_j^{(n)})|,
\]
Corrected radius (trustable),

\[
(14) \quad r_{\text{correct}} = r(n) - r_{\text{error}},
\]
Here \( z_j^{(n)} = r(n) w_j^{(n)} \), lower index \( j \) is the number of roots and \( m \leq n \).

Easily we can do error estimation by whether \( \delta < \) Tolerance Error is satisfied or not for the resolved roots. If there are any resolved roots that do not satisfy this condition, they are rejected. Moreover, resolved roots outside of the polydisc radius of which is defined as distance between the center of convergence and rejected resolved root with (14) are also rejected. This first process decreases complex error estimation with complex error treatment.

2.3 Normalizing requirement for actual execution by finite series approximation

We defined polynomial function for approximation as (12). Applying DKA method to \( \tilde{f}(w) \) in (12), we must normalize it as \( P(z) \) in (1). Then we devide \( \tilde{f}(w) \) by \( b_n \) for normalization. It is important to avoid divergence of coefficients of polynomial function for approximation. We found that variable transform in (12) removes this divergence. If we scamp this treatment, we need large digit for computation. Because \( a_n \) is very small for large \( n \), we need large digit for \( 1/a_n \) for normalization.

3 Examples

3.1 Low order polynomial

As an example, we applied the method to BesselJ0(x) with 20 order approximation for DKA method. Figure 6(a) shows plots of BesselJ0(x) and polynomial function for approximation. Figure 6(b) shows resolved roots. Horizontal line in the center of figure corresponds to real axis. Meshes in this figure show convergence trajectories of resolved roots in each iteration step. We find that 4 roots are inside unit circle at least by numerical resolution.
3.2 Higher order approximation

*BesselJ_0(x)* function and its 40 order approximation are shown in Fig. 7(a) and resolution by DKA is shown in Fig. 7(b). We can obtain more roots at the same time by the higher order approximation method than by the lower order one.

![Image](Fig. 7 (a) Plots of BesselJ0(x) and 40 order approximation function, (b) Numerical result)

3.2 *Cos(x)* by 3rd order method with 80 order approximations

Numerical resolution of periodic function *cos(x)* and its approximation by 80 order polynomial function are shown in Fig. 8(a), and its roots by numerical resolution with 3rd order method are shown in Fig. 8(b). We can find many unnecessary roots outside unit circle. It seems possible that the number of these fictitious roots become smaller by excellent series expansion. By this method, we obtain the same number of resolved roots to the number of order of polynomial function for the approximation, whether we desire it or not.

![Image](Fig. 8 (a) Plots of cos(x) by 80 order approximation function, (b) resolution by the method)

4 Continuation of local solution

We considered local resolution of roots for the holomorphic function up to this point. In the following section, we consider global distribution of the roots by connection and continuation. Figure 9 illustrates global distribution of roots by connection with many local resolutions by DKA. We consider covering of the some region using polydiscs. Each local polydisc overlap compactly without gaps in the region. Resolution of the holomorphic function on each polydisc is possible with proposed method in the previous section. Small circles and triangles in Fig. 9 correspond to roots by the method on each polydisc and Fig.10 shows image of continuation of resolution.

![Image](Fig. 9 Continuation or connection with polydiscs for global resolution of roots)

![Image](Fig. 10 Image of continuation by proposed method)

In this case, we can define and use convergence radius on each polydisc. Therefore, we will obtain reliable roots and fictitious roots by the method in the each polydisc. When a root by the method in a polydisc is not found in another polydisc where the region of the both polydiscs is overlapped, we must consider such root is fictitious one. Moreover, we must identify or distinguish two close roots. The one is obtained in a polydisc and the other is obtained in another polydisc. It is difficult to identify or distinguish such two roots in overlapped region, because both are very close and we have no information of the distribution about the roots in this region. As for the prescription of this difficulty, we use square regions. The size of square regions are defined by the smallest radius of all polydiscs in the
region. Sides of each square region consist of two kinds of boundaries, which are defined as in Fig. 11.

We only allow resolved roots in each square region which are covered by a polydisc that includes the square wholly. Using this connection, continuation and selection, we can obtain all resolved roots without redundancy in the whole region.

5 Application of DKA method with this procedure

We can use above method to approximate \( u(z) \) in (8). To construct unit function \( u(z) \), we treat as following,

\[
\tilde{P}(z) = \prod_{i=1}^{n} (z - z_i^{(\infty)}) = \prod_{i=1}^{n} (z - r(n)w_i^{(\infty)}),
\]

\( i = 1, 2, ..., m \), here \( w_i^{(\infty)}, i = 1, ..., m \) are resolved roots by DKA or 3rd method inside polydisc. \( m (m \leq n) \) is number of roots inside polydisc of which radius is (14). Then we put,

\[
\frac{1}{\tilde{P}(z)} = \frac{1}{\tilde{P}(z)} f(z), \quad u(z) = \tilde{u}(z)
\]

for the approximation of \( u(z) \). Rough estimation of \( u(z) \) gives effective information for the exact treatment of Weierstrass preparation theorem for \( f(z) \), and \( \tilde{P}(z) \) in each polydisc gives algebraic object to represent original holomorphic function by difference (algebraic) formula.

7 Summaries and Conclusions

Extended treatment of DKA and 3rd order method for holomorphic (analytic) functions are proposed. Weierstrass preparation theorem is treated from numerical point of view. Local discrete representations of holomorphic functions are possible using proposed extended method. New numerical treatment for evaluating residues of holomorphic functions will be developed by this approach.

References:


Appendix  Holomorphic function in \( C^n \)

**Definition 1:** A complex-valued function \( f \) defined on \( C^n \) is called *holomorphic function* in \( C^n \) if each point \( w \in C^n \) has an open neighborhood \( U \), \( w \in U \subset C^n \), such that the function \( f \) has a power series expression

\[
\sum_{v} a_{v} (z_{1} - w_{1})^{v_{1}} \cdots (z_{n} - w_{n})^{v_{n}}
\]

which converges for all \( z \in U \). Here \( C^n = C \times \cdots \times C \) which is Cartesian product of \( n \) copies of complex plane, \( x_j \) and \( y_j \) are real numbers and \( z_j = x_j + y_j \in C, z = (z_1, \cdots, z_n) \).

**Definition 2:** An open polydisc in \( C^n \) is a subset \( \Delta(w; r) \subset C^n \) of the form

\[
\Delta(w; r) = \Delta(w_1, \cdots, w_n; r_1, \cdots, r_n) = \{ z \in C^n : |z_j - w_j| < r_j, 1 \leq j \leq n \}
\]

the point \( w \in C^n \) is called the *center* of the polydisc, and \( r = (r_1, \cdots, r_n) \in R^n, r_j > 0 \) is called the polyradius.

Note that polynomials in the functions \( z_j, \ldots, z_n \) are holomorphic in all \( C^n \). It is familiar result from elementary analysis that a power series expansion of the form (A.1) is *absolutely uniformly* convergent in all suitable small open polydiscs \( \Delta(w; r) \) centered at the point \( w \). Moreover, the function \( f \) is holomorphic in each variable separately throughout the domain in which it is analytic.