The Analyses and Applications of the Traffic Dispersion Model

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Abstract: - In this study, we discuss the derivation, applications and the necessity of the traffic dispersion model, which is a nonlinear Poisson equation. Also, the analyses are presented in the content. The model is derived from the interaction among vehicles on a road. Therefore, the traffic pressure can be described by the model and the relation between density and the traffic pressure, which is transformed into the traffic field in this study actually. By the dispersion model, density on a multilane road can be distributed according to the interaction that makes the discussion of multilane traffic easy. If there are multiple types of driving behaviors, the density and the interaction can also be related by the model under the finite-space requirement, which describes all kinds of vehicles share the same road section. Thus, the closure relation of dynamic traffic model is obtained by the dispersion model with equilibrium relations. Furthermore, the analytical solution of linear model and analyses of the nonlinear model are presented in this study.

Key-Words: - Traffic Flow, Poisson Equation, Self-Consistent System, Multiclass Users, Multilane Traffic.

1 Introduction

Traffic flow researches are generated by the worsening of urban traffic congestion, especially in the recent two decades. According to the history of traffic flow researches, modeling approaches evolve from static models to dynamic models. The main applications of the static models are planning and design. On the other hand, the main applications of the dynamic models are traffic control and prediction of dynamic information. In the development of the dynamic traffic flow, the car-following theory [1,2] and the kinetic wave theory [3,4] are the first two dynamic modeling approaches. After that, Boltzmann-like model [5], vehicular gas-kinetic model [6, 7] and cellular automation [8] are presented.

Since most roads are multilane in the real world, traffic flow researches are extended from single lane models to multilane models. Car-following theory is extended to multilane traffic by Wiedemann [1], who incorporated lane-changing and overtaking in his modeling approach. His modeling approach becomes the foundation of many emporary microscopic multilane models. Compressible fluid models are also extended for multilane freeway on-ramp perturbation studies by Munjal and Pipes [9]. In these studies, the rate of lane-change is hypothesized as an oscillation around the difference of equilibrium densities between adjacent lanes. In the studies of lane-changing behavior, work using mathematical modeling had been done by Munjal, Hsu and Lawrence [10], Michalopoulos, Beskos, and Yamauchi [11] and etc. Hoogendoorn and Bovy [7] have developed a multilane multiclass traffic flow model based on mesoscopic principles. The model inherits a number of the properties of the gas-kinetic equations (e.g. description using platoons, finite-space requirements). Cho and Lo [12] also presented a dynamic multiclass multilane traffic flow model by a similar deriving procedure. The systematic model includes continuity, motion and variance equations, so as to describe the evolution of the traffic flow. The state of the traffic flow is determined by the dispersion model, which is a nonlinear Poisson equation.

In this study, we discuss the applications and the analyses of the dispersion model in detail. There are several important applications of the dispersion model. Firstly, the model is employed to describe the state at each time point. As mentioned above, the multilane traffic model is a difficult problem. The dispersion model provides a simple modeling approach. Secondly, the dispersion model also implies the finite-space requirements. If there are multiple types of driving behaviors, the model can be employed to evaluate the aggregated influence of density. This advantage is useful when vehicles are not restricted to drive along one by one in a single lane, such as motorcycles and bicycles. Another importance of the dispersion model is the model relates the basic variables in traffic flow and makes the systematic equations self-consistent [12]. The analytical solution and analyses are also show in this study.

The rest of this paper is organized as follows. Section 2 presents the derivation of the linear dispersion model briefly. Section 3 introduces the analytical solution of the model. Section 4 extends the model to the nonlinear Poisson equation and its solution method. In section 5, the applications of the model for traffic flow are shown. After that, the paper concludes with some perspectives in section 6.

2 Derivation of the Dispersion Model

In this section, the dispersion model is derived in brief. A multilane road is considered as a two-dimensional domain, herein. Such a consideration simplifies the discussion of multilane traffic behavior and makes the development of multilane models much easier. Therefore, we review the definition of the three basic traffic variables in two-dimensional space at first. The variables are defined in the rectangular Cartesian coordinate system. Flow [$pcu\times time^{-1}$] denotes the number of vehicles (or passenger car unit, pcu) passing through a certain section during a given time period. Flow is a scalar and is denoted by Q. Density [$pcu\times length^{-2}$ or $pcu\times length^{-1}$ *per lane*] is the number of vehicles (pcu) occupying a section of road. Density is a scalar and is denoted by k. Velocity [$length\times time^{-1}$] denotes the length that vehicles passing through during a specified time period. Velocity is a vector and is denoted by $\mathbf{u}=(u_x,u_y)$, where u_x and u_y means speed in x- and y-direction, respectively. $\|\mathbf{u}\|=(u_x^2+u_y^2)^{1/2}$. A new variable flow density [$pcu\times(length\times time)^{-1}$] which denotes the number of vehicles (pcu) passing through a unit width of a road in a given time period is introduced. Flow density is a vector and is denoted by $\mathbf{q}=(q_x,q_y)$, where q_x and q_y means flow in x- and y-direction, respectively. q_y can only be zero, if there is no entrance or exit from the roadside. $\|\mathbf{q}\|=(q_x^2+q_y^2)^{1/2}$. $\mathbf{q}=k\mathbf{u}$, where $k\mathbf{u}=(ku_x,ku_y)=(q_x,q_y)$.

The dispersion model is based on the concept of traffic field, which is extended from car-follow theory. Car-following theory [2, 13, 14] describes that each driver reacts mainly to a stimulus from his immediate environment according to the relationship as (Reaction) $_{t+T}$ = λ (Stimulus) $_t$, where λ is a sensitivity coefficient and T is a reaction time-lag. The general car-following model is

$$x''_{n+1}(t+T)=\frac{a(x'(t+T))^m_{n+1}[x'_n(t)-x'_{n+1}(t)]}{[x_n(t)-x_{n+1}(t)]^l}, \qquad (1)$$

where $x_{n+1}(t)$ denotes the position of the $n+1$ th vehicle at time t, x' and x'' are its velocity and acceleration. Equation (1) can be treated as Newton's law, which implies that acceleration is induced by external force. On the other hand, the external force can also be described by field, especially the isolated force. For the sake of safety, one vehicle on a road adjusts its velocity and spacing according to the relative position between others cars so as to avoid accidents. It is assumed that each vehicle has its own field. As mentioned above, the traffic pressure (or traffic force), which is produced by the interaction among vehicles, is a resistance. On the other hand, the vehicles are assumed to be pushed forward by a traffic force on the boundary. Thus, the total force of thrust and the resistant force determine the velocity and acceleration of vehicles on the road. By Newton's law, traffic field $\tilde{\mathbf{E}}$ is in proportion to acceleration $d\mathbf{v}/dt$, where $\mathbf{v}$ is individual velocity. From the discussion of car-following theory, it can also be assumed that traffic field ($\tilde{\mathbf{E}}$) depends on velocity and headway. To simplify the complication of the problem, $\tilde{\mathbf{E}}$ is assumed to depend on spacing and to satisfy the inverse-square law. It is reasonable to assume that the influence of cars in the same lane is larger than that in the adjacent lanes. Let the influence of cars in the same lane be M times larger than that in the adjacent lanes. If we consider the interaction between two vehicles (vehicle 0 and 1), the traffic field produced by vehicle 1 (leader) will act on vehicle 0 (follower). The traffic field acting on vehicle 0 can be formulated as:

$$\tilde{\mathbf{E}}_{01}=\frac{e}{\varepsilon_0}\left(\frac{\tilde{x}_0-\tilde{x}_1}{\left|\tilde{x}_0-\tilde{x}_1\right|^3}\mathbf{i}+\frac{\tilde{y}_0-\tilde{y}_1}{M^2\left|\tilde{y}_0-\tilde{y}_1\right|^3}\mathbf{j}\right), \qquad (2)$$

where e is the passenger car equilivalent, ε_0 is the interacting parameter, $(\tilde{x}_0,\tilde{y}_0)$ and $(\tilde{x}_1,\tilde{y}_1)$ are the positions of vehicle 0 and 1, respectively. The influence between two vehicles is larger as the distance between them is smaller in the real traffic stream. Therefore, the assumption of inverse-square law is reasonable herein. For the sake of convenience, we transform the domain from $\tilde{\Omega}$ to Ω, that is, let $x=\tilde{x}, y=M\tilde{y}$ and traffic field acting on vehicle 0 in Ω is denoted by

$$\mathbf{E}_{01}=\frac{e}{\varepsilon_0}\frac{\mathbf{X}}{\|\mathbf{X}\|^3}, \qquad (3)$$

where $\mathbf{X}$ denotes the distance vector from vehicle 0 to vehicle 1. Under the assumption of superposition, the relation between two vehicles can be extended to the whole road section. The traffic field acting on vehicle 0 is represented as

$$\mathbf{E}=\sum_{i}^{N}\left(e_i\mathbf{X}_i/\varepsilon_i\|\mathbf{X}_i\|^3\right), \tag{4}$$

where N is the number of vehicles that may interact with vehicle 0, $\mathbf{X}_i$ denotes the spacing from vehicle i to vehicle 0. In the continuous space, equation (4) can be represented as

$$\mathbf{E}=\frac{e}{\varepsilon}\int_{\Omega}\left((k-k_s)/\|\mathbf{X}\|^2\right)d\Omega, \tag{5}$$

where Ω is the road section and ε denotes the interacting parameter. k is the density and k_s is the unrestrained density under the given condition. Then, a potential function ϕ exists by the potential theory. Because the traffic field is a conservative field, which induces that the traffic potential of a vehicle depends on its relative position only and the potential is independent of its moving path. This result makes us to discuss multilane traffic easy. If we have to consider the possible paths of a vehicle, the problem will become more complicated than considering the position only. The potential function ϕ satisfies $\mathbf{E}=-\nabla_x\phi$. Thus, the magnitude of traffic field is illustrated as

$$div\mathbf{E}=-\Delta\phi=e(k-k_s)/\varepsilon+K_a, \tag{6}$$

where $div\mathbf{E}$ denotes the magnitude of traffic field, $K_a=K_a(x)$, which depends on the position x, is the adjust term of the road condition if the road condition is ideal $K_a=0$.

3 Analytical Solution of the Linear Dispersion Model

The dispersion model needs to couple with a set of boundary conditions to make a model complete. Since vehicles move forward along a specific direction, the boundary condition of the model should be given in order to consist with the situation. If vehicles move from left to right, the left boundary condition must be larger than the right boundary condition. Because vehicles moving along the direction of traffic field, which is directed from high potential to low potential. However, the traffic potential cannot be measured directly. The boundary conditions have to be transformed from density by equation (6). If a model with a set of boundary conditions are given as follows:

$$\begin{cases}\Delta\phi(x,y)=-e(k-k_s)/\varepsilon+K_a & \text{in }\Omega\\ \dfrac{\partial\phi}{\partial n}(x,0)=\dfrac{\partial\phi}{\partial n}(x,L_y)=0 \quad 0\le x\le L_x & \text{on }\partial\Omega\\ \phi(0,y)=g(y),\quad \phi(L_x,y)=f(y)\quad 0\le y\le L_y & \text{on }\partial\Omega\end{cases}, \tag{7}$$

where L_x and L_y are the length of the research doamin in the x- and y-direction. The boundary conditions mean that there is no entrance and exit on the both roadsides. In addition, inflow density, velocity, outflow density and velocity are controlled by specific function, such as signal or speed limited. Solving the following three sub-problems, we will obtained the solution of problem (7) by superposing the three results.

$$\begin{cases}\Delta\phi(x,y)=0 & \text{in }\Omega\\ \dfrac{\partial\phi}{\partial n}(x,0)=\dfrac{\partial\phi}{\partial n}(x,L_y)=0 \quad 0\le x\le L_x & \text{on }\partial\Omega\\ \phi(0,y)=g(y),\quad \phi(L_x,y)=0\quad 0\le y\le L_y & \text{on }\partial\Omega\end{cases}, \tag{8}$$

$$\begin{cases}\Delta\phi(x,y)=0 & \text{in }\Omega\\ \dfrac{\partial\phi}{\partial n}(x,0)=\dfrac{\partial\phi}{\partial n}(x,L_y)=0 \quad 0\le x\le L_x & \text{on }\partial\Omega\\ \phi(0,y)=0,\quad \phi(L_x,y)=f(y)\quad 0\le y\le L_y & \text{on }\partial\Omega\end{cases}, \tag{9}$$

$$\begin{cases}\Delta\phi(x,y)=-e(k-k_s)/\varepsilon+K_a & \text{in }\Omega\\ \dfrac{\partial\phi}{\partial n}(x,0)=\dfrac{\partial\phi}{\partial n}(x,L_y)=0 \quad 0\le x\le L_x & \text{on }\partial\Omega\\ \phi(0,y)=0,\quad \phi(L_x,y)=0\quad 0\le y\le L_y & \text{on }\partial\Omega\end{cases}. \tag{10}$$

The problems can be solved by separation of variables. Solution problem (8), (9) and (10) are shown as equation (11), (12) and (13), respectively.

$$\phi(x,y)=B_0(1-x)+\sum_{n=1}^{\infty}B_n\sinh n\pi(1-x)\cos(n\pi y), \tag{11}$$

where B_i , for i=0,1,…, are determined by the Fourier coefficients of *g(y)*.

$$\phi(x,y)=f_0x+\sum_{n=1}^{\infty}\frac{\sinh n\pi x}{\sinh(n\pi)}\cos(n\pi y), \tag{12}$$

where f_0 is determined by the Fourier coefficients of *f(y)*.

$$\phi(x,y)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{-k_{mn}}{(m^2+n^2)\pi^2}\sin n\pi x\cos m\pi y, \tag{13}$$

where k_{mn}, for m=0,1,…, n=0,1,…, are determined by the Fourier coefficients of $e(k-k_s)/\varepsilon+K_a$.

Therefore, the general form of the analytic solution is shown as

$$\phi(x,y)=A_0+\sum_{i=1}^{n}A_i x^n+\sum_{i=1}^{n}B_i y^n+ \sum_{i=0}^{n}\sum_{j=0}^{m}\left[C_i\sinh(n\pi x/L_x)+D_j\cosh(m\pi y/L_x)\right. \left.+E_i\sin(n\pi x/L_x)+F_i\cos(n\pi y/L_y)\right]\left(\sin(n\pi x/L_x)+\cos(n\pi y/L_y)\right) \quad ,(14)$$

where A_i, B_i, C_i, D_i, E_i and F_i are coefficients, which are determined by the boundary conditions and the explicit form of density function. Also, the dispersion model can be solved by the numerical methods such as finite difference methods, finite elements and finite volume methods. typical cases are illustrated as follows: The existence and uniqueness can be proved by the maximum principle.

4 Nonlinear Dispersion Model

The vehicular dispersion model derived in the previous section is a linear Poisson equation, since density only depends on position and time. The dispersion model implies several facts. The first one is that the magnitude of traffic field is large as density is large. The second one is that the direction of field is from high protential to low potential. As a matter of fact, density is distributed by traffic field, which is induced by traffic potential. However, the linear traffic dispersion model describes that potential is determined by density function and density is independent of potential; that is, $\phi=\phi(\mathbf{x},k)$, $k=k(\mathbf{x},t)$. As we know that density and potential should depend on each other; that is $\phi=\phi(\mathbf{x},k)$, $k=k(\mathbf{x},t,\phi)$. Because of the reason, a nonlinear dispersion model is presented. We assume that the density will tend toward its equilibrium distribution, which is the most possible microscopic state under a specific macroscopic situation. Hence, the equilibrium distribution is derived from a mathematical programming, whose objective is finding out the most possible microscopic state under the specific macroscopic phenomena given by the constraints. The mathematical programming is shown as follows:

$$Max \quad W=\frac{N!}{\prod_i n_i!}, \qquad \forall i\in\Omega \quad (15)$$

$$s.t. \quad \sum_i n_i=n_1+n_2+\cdots+n_m=N, \qquad \forall i\in\Omega \quad (16)$$

$$\sum_i n_i\Theta_i=n_1\Theta_1+n_2\Theta_2+\cdots+n_m\Theta_m=\Theta_{tol}, \forall i\in\Omega \quad (17)$$

where i is the number of intervals, N is the number of total vehicle, n_i is the vehicle number of interval i, Θ_{tol} is the total velocity variance, and Θ_i is the velocity variance in invertal i. The velocity variance of individual car is defined as $\left\|\bar{\mathbf{u}}_i-\mathbf{u}_e\right\|^2$, where $\bar{\mathbf{u}}_i$ is the average velocity of interval i and $\mathbf{u}_e$ is the equilibrium velocity. Equation (15) is the objective function, and equation (16)~(17) are given macroscopic phenomena. Equation (15) finds the most possible number of combinations of n_i if the total number of cars on the road is N. Equation (15) is in a simplified form, which neglects the all-possible combinations of the denominator, since the denominator is a constant. Equation (16) is the conservation of vehicle numbers, and equation (17) is the conservation of total velocity variance. The mathematical programming (15)~(17) can be solved by the following steps. First, use Stirling's equation to approximate equation (15), and $\ln W=N\ln N-\sum_i n_i\ln n_i$ is obtained. From the first order condition, when W is the maximum, $\partial W_{\max}/\partial n_i=0$ is true, which implies $\partial\ln W_{\max}/\partial n_i=0$. Then, by KKT condition, we have

$$n(\Theta)=N/\Theta_e\exp(-\Theta/\Theta_e). \quad (18)$$

The traffic variable concerned in this study is density (k). Let the essential density K_0 be $N\pi/\Theta_e h^2$, mobile density is

$$k(\Theta)=K_0\exp(-\Theta/\Theta_e). \quad (19)$$

Equation (19) describes that as velocity variance increases, density decreases, which implies that vehicles with large velocity variance can spread out easily but induces unstable traffic flow at the same time. However, in the real traffic condition, the mobile density does not spread out immediately as the velocity variance increases. There exists a threshold Θ_0. When the velocity variance is larger than Θ_0, the mobile density will be less than the essential density; otherwise the mobile density will be larger than the essential density. The modified model is

$$k(\Theta)=K_0\exp(-(\Theta-\Theta_0)/\Theta_e). \quad (20)$$

Therefore, vehicles are distributed on the road by equation (20). In traffic flow study, not all traffic conditions need to consider velocity variance. Some simple cases only need to consider density or density and velocity at the same time. For this reason, converting velocity variance to traffic potential is necessary. From the relation among the traffic field, traffic pressure, traffic potential and velocity variance, equation (20) becomes

$$k=K_0\exp(e\psi-e\phi/\Theta_e), \quad (21)$$

where ψ is the potential equivalent of the velocity variance threshold Θ_0; that is, $\psi = -\Theta_0/e$. ψ is named as potential barrier here. Equation (21) gives the equilibrium distribution of density and implies several facts. The first one is that density is decreasing as traffic potential is increasing. The second one is as the equilibrium velocity variance increases, the variation of density increases, which means the traffic is sensitive. The third one is as the potential barrier is low, the density is small. That is, drivers are aggressive. They can spread out easily although the velocity variance is small. Figure 1 illustrates the three aspects. Coupling equation (6) to (21), the nonlinear dispersion model is given as

$$-\Delta\phi = \frac{e}{\varepsilon}\left[K_0 \exp\left(-\frac{e\phi - e\psi}{\Theta_e}\right) - k_s\right] + K_a, \qquad (22)$$

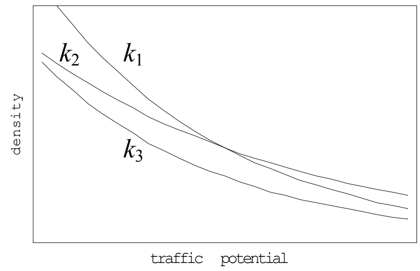

Fig. 1 The density curves of different potential barrier and equilibrium velocity variance, where $k_1 = K_0\exp((e\psi_1 - e\phi)/\Theta_{e1})$, $k_2 = K_0\exp((e\psi_1 - e\phi)/\Theta_{e2})$ and $k_3 = K_0\exp((e\psi_2 - e\phi)/\Theta_{e1})$. $\Theta_{e1} > \Theta_{e2}$ and $\psi_1 > \psi_2$.

The nonlinear model

$$\iint div\nabla\cdot\phi dxdy = \iint\left[\frac{e}{\varepsilon}\exp\left(\frac{e\psi - e\phi}{\Theta_e}\right) - \frac{e}{\varepsilon}k_s + K_a\right]dxdy \cdot \qquad (23)$$

By central difference method, we can derive a direct computation equation (10).

$$\left(\frac{\phi_{i+1,j} - \phi_{i,j}}{hx_i}\right) + \left(\frac{\phi_{i-1,j} - \phi_{i,j}}{hx_{i-1}}\right) + \left(\frac{\phi_{i,j+1} - \phi_{i,j}}{hy_j}\right) + \left(\frac{\phi_{i,j-1} - \phi_{i,j}}{hy_{i-1}}\right)$$
$$= \left(\frac{hx_i + hx_{i-1}}{2}\right)\left(\frac{hy_j + hy_{j-1}}{2}\right) \times \left[\frac{e}{\varepsilon}\exp\left(\frac{e\psi_{i,j} - e\phi_{i,j}}{\Theta_e}\right) - \frac{e}{\varepsilon}k_{si,j} + K_{ai,j}\right], \qquad (24)$$

where hx_i is the length of the interval *i* in *x*-direction and hy_j is the length of the interval *j* in *y*-direction

5 Applications for Traffic Flow

Solving the model with boundary conditions, the potential function is determined. Thus, the traffic field is obtained by equation (6) and density is obtained by equation (21). From the traffic field, we can observe how traffic pressure acts on vehicles driving along the road. Also, the variation of velocity variance can be observed. With traffic force, velocity variance and density, the multilane traffic behavior can be described. Figure 2 is an example of a basic section of freeway [15], which is behind an on-ramp. Vehicles tend toward the inside lane and the trend of the vehicles in the outside lane is stronger than that of the vehicles in the median lane. That is, there are more vehicles trying to change lanes in the front part of the outside lane. However, the lane-changing trend keeps a longer distance in the median lane. If the uninterrupted section of the freeway is long enough, the traffic flow will become uniform. The information is also useful to design the location of the interchanges of the freeway so as to optimize the traffic volume.

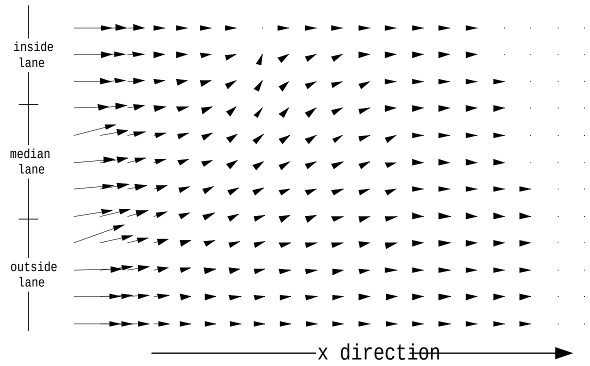

Figure 2. The traffic field of the numerical example, which shows that vehicles tend to move to the inside lane and the traffic flow will become uniform.

The system can be extended to multi-class users model by employing the concept of Hoogendoorn and Bovy [7]. They considered that each class has different behavior, and share the same limited space. The concept can be applied to Poisson equation easily. If there are *i* classes of users or vehicles on the road, equations (6), (21) are modified as equations (25)~(26) and the aggregated density is given by equation (27).

$$div\mathbf{E} = -\Delta\phi = \sum_i \frac{e_i k_i}{\varepsilon_i} - \frac{e}{\varepsilon}k_s + K_a, \qquad (25)$$

$$k_i = K_{i0}\exp\left(-\frac{e_i\phi - e_i\psi_i}{\Theta_{ie}}\right), \qquad (26)$$

$$k(\mathbf{x},t) = \sum_i e_i k_i(\mathbf{x},t) \cdot \qquad (27)$$

Equations (25)~(27) are very useful as vehicles are not restricted to move along a single, such as the motorcycles in Taiwan. The most important application of the dispersion model is making a dynamic system self-consistent. A solving procedure of a self-consistent dynamic system is illustrated in figure 4 [12]. The concept of modeling can handle the different characteristics and distribute density on a road.

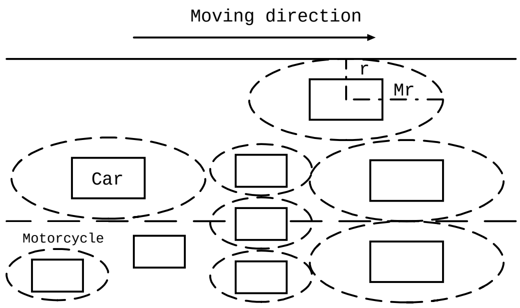

Fig. 3 motorcycles and cars mixed flow

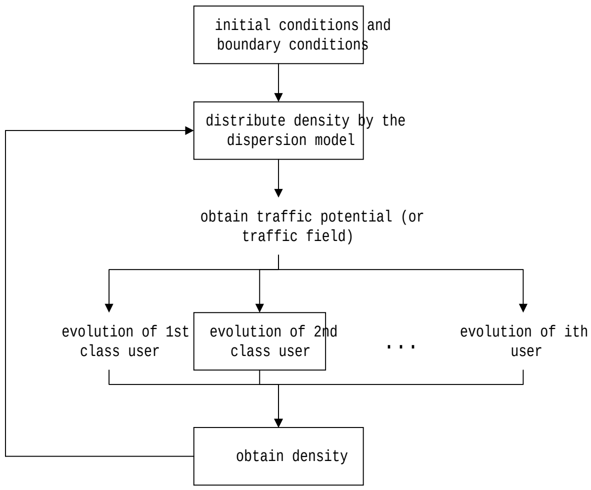

Fig. 4 Self-consistent traffic flow modeling

6 Conclusions and Perspectives

Mostly, Poisson equation is employed to describe the diffusive phenomena, such as heat and density, in physical researches. It is introduced into traffic flow research to explain the dispersion of vehicles in this paper. Traffic flow theory is the fundamental research of traffic science. In this study, the analyses and applications are presented. The model can be employed to analyze multilane, multiclass users traffic. The most important application is coupling the dispersion model with kinematic models and consists a self-consistent system. The linear dispersion model is solved by the analytical solution and the nonlinear dispersion model must be solved by a numerical scheme. The researches of numerical analyses are left for further researches.

Acknowledgments

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