Attraction Basins of Non-linear Differential Equation Set: Determination of Initial Values of Phase Variables for Correct Integration of the System

P.P. HORLEY[a,b], P.YU. VOROBIEV[c], V.V. GORLEY[b]

YU.V. VOROBIEV[a] and J. GONZÁLEZ-HERNÁNDEZ[a]

[a] CINVESTAV-IPN, Unidad Querétaro, Juriquilla, Querétaro, Qro., MÉXICO

[b] Chernivtsi National University, Chernivtsi, UKRAINE

[c] Facultad de Ingeniería, Universidad Autónoma de Querétaro, Qro., MÉXICO

Abstract: - New application proposed for the classic Julia set algorithm to determine the attraction basins of non-linear systems of differential equations for self-organizing systems, providing the information of proper choice of initial values of phase variables and their dependence on the control parameters. The method proposed was successfully applied for different models of self-organized processes in semiconductor physics.

Key-Words: - non-linear systems, self-organization, phase variables, integrability, attraction basin

1 Introduction

Non-equilibrium and non-linear processes in physics, chemistry, biology, economics and other fields of science are being extensively investigated nowadays using self-organization methodology [1-3], making it possible to obtain the qualitative description of the system evolution, predict the influence of the external control parameters, etc. The mathematical models of the systems mentioned mainly are the sets of ordinary differential equation (ODE) for the phase variables. The analytical solution of them is quite complicated and in general is not possible; numerical methods developed allows to obtain the solution; though, the latter depends significantly on the proper choice of initial values of phase variables and the integration step. In the given paper we have proposed the simple method to define the possible ranges of the initial values of phase variables to ensure the integrability of the system and to investigate their evolution with changes of control parameter.

2 Attraction basin for ODE system

Let us consider the mathematical model of the self-organization process to be given as the set of differential equations for n phase variables $y_1 \ldots y_n$ over the independent variable x and with m control parameters $c_1 \ldots c_m$

$$\frac{dy_n}{dx} = f_n(x, y_1, y_2, \ldots y_n, c_1, c_2, \ldots c_m) . \quad (1)$$

The set (1) is usually solved with numerical one-step methods, such as methods of Euler or Runge-Kutta [4]. They have good precision and are quite quick. If the better precision should be reached, one needs to use prognosis and correction methods to reduce the numeric error [4]. Any of the numeric methods named is influenced strongly by the proper value of integration step and proper choice of initial values of phase variables. Of course, smaller integration steps give us more stable integration process, as the changes of the function could more accurately track the changes of independent variable, but in this case we need to consider more points to reach the certain value of independent variable; if the integration step becomes greater than the characteristic exponents of the system, the system will not integrate properly and diverges.

The initial phase variable values are also the question of proper choice. If they could not be chosen from the physical reasons or the boundary conditions, one needs to assign them the values to ensure integrability of the system and to prevent its divergence. If we mark all the suitable for integration initial values of the phase variables, then for n-dimensional system we will obtain n-dimensional volume in the phase space. Its shape is characteristic for the given system under given ratio of control parameters and depends on it; therefore if one could

find the method to define the boundary between converging and diverging states in the phase variable space, it would be possible to ensure the integration of the system. Moreover, observing the changes of initial values of phase variables will allow us to determine their threshold values, when self-organization processes for the model considered take place.

The simple and accurate solution of the problem mentioned could be obtained applying the standard Julia set algorithm [5, 6] to ODE set. Classically Julia set is built for the iteration sequence

$$\mathbf{Z} = \mathbf{Z}^2 + \mathbf{C}, \tag{2}$$

where **Z** and **C** are complex numbers. While iterating (2) one can face two possibilities: a) for the given set of initial conditions $\mathbf{Z}_{\mathrm{Re}}^0$ and $\mathbf{Z}_{\mathrm{Im}}^0$ the iteration sequence tends to infinity, increasing the value of **Z** with each iteration; b) the values of $\mathbf{Z}_{\mathrm{Re}}$ and $\mathbf{Z}_{\mathrm{Im}}$ during iteration process are being attracted to some periodical orbit for the given pair of initial $\mathbf{Z}_{\mathrm{Re}}^0$ and $\mathbf{Z}_{\mathrm{Im}}^0$. Substituting in the above sequence "iteration of the sequence" to "integration of ODE set" one could find the new application of the method described. Indeed, if the system diverges, it means that the phase points are being attracted to infinity; otherwise they are being attracted to the attractor – phase portrait of the system (focus, limit cycle, strange attractor, etc.) that is the solution of ODE set. The boundary between the two types mentioned of the points in the phase space defines the attraction basin of the system.

To represent the degree of the divergence of the system we can use the length of phase radius-vector *R* either from point y_i^0 or from the reference point of the system:

$$R_{init} = \sqrt{\sum_{i=1}^{n} (y_i - y_i^0)^2}\,, \tag{3}$$

$$R_{zero} = \sqrt{\sum_{i=1}^{n} y_i^2}\,. \tag{4}$$

If the value of *R* becomes greater in some orders of magnitude that the expected value for the system studied – let us say, some given value R_{MAX}, it will mean that the system diverges. The formula (4) can be used for the relatively small initial values of phase variables, otherwise one could set R_{MAX} to be sufficiently big to ensure $R_{MAX}^2 >> \sum_{i=1}^{n} \left(y_i^0\right)^2$, or the system due to the algorithm will be considered unintegrable even before the integration process; though the greater values of R_{MAX} will cause big calculation time. To avoid this difficulties one should use formula (3) in this case, considering relative changes of the *R*. To make the convergence/divergence chart easier to read, we can use the common methodology that was used for Julia sets: in the case of divergence the color intensity of the point for the given y_i^0 will refer to the number of the integration step, when the radius-vector R_{init} or R_{zero} becomes greater than R_{MAX}. To mark the convergence region we will hatch it.

3 Results and discussion

The attraction basins for ODE set were successfully used to define the initial values of phase variables for different systems from the semiconductor physics; in this paper we will present the results obtained for photo-induced Gunn effect in semiconductor with GaAs parameters [7, 8]. The system is being described with the set of two ODEs

$$\begin{cases} \dfrac{d\mathrm{n}}{dx} = \dfrac{1}{\beta}\left[J_c - \mathrm{n}v(\mathrm{E})\right] \\ \dfrac{d\mathrm{E}}{dx} = \dfrac{1}{\alpha\beta}\left[\dfrac{ab(1+m\cos kx)}{\mathrm{n}+a(1+m\cos kx)} - \mathrm{n} - 1\right] \end{cases}, \tag{5}$$

where n and E are dimensionless concentration of electrons and electric field distribution along the sample, J_c is equal to density of electric current in the external circuit; external illumination parameters is given by parameter *a*, *k* is the wave vector and *m* is modulation depth of the incident light, α, β and *b* are the constants depending on the material properties.

Using our methodology we have defined the possible initial values of the system – see Fig. 1. The central oval part is the integrability area; outside it the system diverges. Its brighter colors correspond to fast divergence, while the darker – to slower one. As the attraction basin of the system is situated close to the starting point of the reference system, we have used the formula (4) for our calculations with value of R_{MAX}=100 for 8000 integration steps for each point.

In the Fig. 2 we have presented the dependence of the initial electron concentration values upon the changes of external control parameters, such as acceptor concentration N_a and incident light intensity I_0, which modify control parameters *a, b,* α and β.

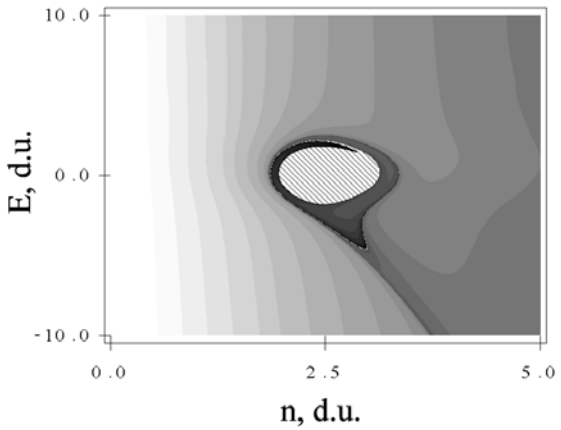

FIG.1. The attraction basin of the system (5) in (n, E) plane. Hatched area marks the proper initial values of the system, when it is integrable. The shades of gray refer to divergence of the system: lighter colors – fast divergence; darker – slower one.

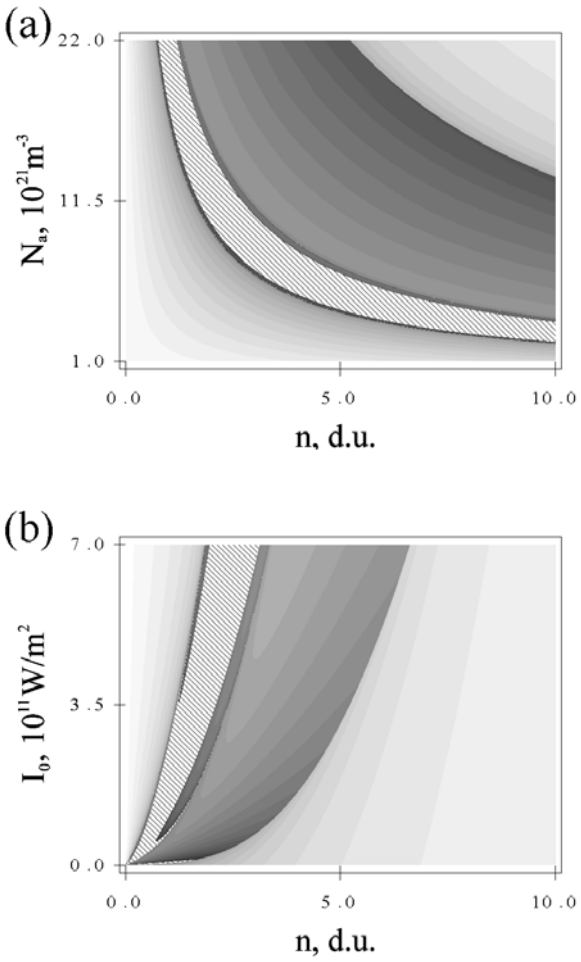

FIG.2. Changes of integrability area for variable n with changes of control parameters. Integrability is denoted in the same way as in Fig. 1.

One can see (Fig. 2a) that increase of acceptor concentration N_a makes our system integrable for the lower values of n. Increase of the light intensity I_0 (Fig. 2b) widens the integrability range, simultaneously shifting it to higher values of n. From the Fig. 2 the threshold values of phase variables could be obtained to initiate the self-organization process for the given ratio of control parameters.

4 Conclusion

In the given paper we have proposed the new application of Julia set building algorithm to non-linear systems of ordinary differential equations. The method was successfully applied to various self-organizing systems, providing us with information about the correct initial values of phase variables to ensure integrability of the system as well as the dependence of these values on the control parameters of the system. The method proposed could be widely used for different models of self-organizing systems in physics, chemistry, biology, etc.

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