Exact Solutions for Coherent Modes of Propagation-Invariant Optical Field

ANDREY S. OSTROVSKY
Facultad de Ciencias Físico Matemáticas
Universidad Autónoma de Puebla
C. P. 72570 Puebla, Pue
MÉXICO

GABRIEL MARTINEZ-NICONOFF, JULIO C. RAMIREZ -SAN-JUAN
Departamento de óptica
Instituto Nacional de Astrofísica, Óptica y Electrónica
C. P. 72000 Puebla, Pue.
MÉXICO

Abstract: The exact solution of partial differential equation for coherent-mode propagation of optical field are obtained under the condition of the statistical invariance in propagation direction and for different types of assigning the mode orthonormality. The examples of propagation-invariant fields of different classes are considered.

Key-Words: Propagation-invariant field; Cross-spectral density function; Mercer expansion; Fredholm integral equation; Coherent-mode structure

1 Introduction

Not so long ago a new interesting class of stochastic optical fields characterized by the invariance of their statistical properties in the direction of propagation has been reported [1]. Recently, we investigated such fields in terms of the coherent modes of oscillations [2]. In this paper we show that the coherent modes of propagation-invariant field can be defined with general solution of the Helmholtz equation considered in the paraxial approximation. The partial solutions of this equation, obtained for different situations of mode orthonormality, define three different classes of propagation-invariant fields. Some interesting examples of the fields representing these classes are considered.

2 Problem formulation

As is well known, a stationary quasimonochromatic optical field propagating in the positive *z* direction is completely characterized by the cross-spectral density function

$$W(\mathbf{x}_1, \mathbf{x}_2; z) = \left\langle V^*(\mathbf{x}_1; z) V(\mathbf{x}_2; z) \right\rangle, \qquad (1)$$

where $V(\mathbf{x}; z)$ is the complex envolvent of field oscillations at a point $\mathbf{x} = (x,y)$ in any plane $z = const$, the angular brackets represent the statistical average taken over the ensemble, and the asterisk denotes the complex conjugate. According to Wolf´s theory [3], the cross-spectral density function of the field localized in some finite domain *S* of the plane *z* may be represented in the form of the Mercer expansion, i.e.,

$$W(\mathbf{x}_1, \mathbf{x}_2; z) = \sum_{n=0}^{\infty} \lambda_n \Phi_n^*(\mathbf{x}_1; z) \Phi_n(\mathbf{x}_2; z), \qquad (2)$$

where λ_n and $\Phi_n(\mathbf{x}; z)$ are the eigenvalues and the eigenfunctions, respectively, of the homogeneous Fredholm integral equation,

$$\int_S W(\mathbf{x}_1, \mathbf{x}_2; z) \Phi_n(\mathbf{x}_2; z) d\mathbf{x}_2 = \lambda_n \Phi_n(\mathbf{x}_1; z). \qquad (3)$$

Each of summands in Eq. (2) represents the cross-spectral density of a field that is completely coherent in the space-frequency domain. Moreover, if the eigenfuction $\Phi_n(\mathbf{x}; z)$ form an orthonormal set (if it is not already so, this may be done by using the Gram-Schmidt procedure), it obeys the same propagation law as the cross-spectral density $W(\mathbf{x}_1, \mathbf{x}_2; z)$, and hence may be regarded as being associated with a mode of the field. For these two reasons, the expression (2) is referred to as coherent-mode representation of the cross-spectral density function.

The latter is an essential tool in describing optical fields of different classes. In fact, any given (subject to the accuracy of some parameters) set of orthonormal functions $\Phi_n(\mathbf{x};z)$ determines a class of the fields which possess certain common features, just as any given (converging) set of values λ_n specifies a certain form of cross-spectral density of the field belonging to this class. In this sense, we refer to the set of functions $\Phi_n(\mathbf{x};z)$ as a coherent-mode structure of the field.

The propagation-invariant field is characterized by a cross-spectral density function which does not change its form in propagation direction, i.e.,

$$W(\mathbf{x}_1,\mathbf{x}_2;z)=W(\mathbf{x}_1,\mathbf{x}_2;0), \qquad z\geq 0. \tag{4}$$

In Ref. [1] the exact solution of the coupled Helmholtz equation for cross-spectral density propagation, obtained under condition (4), is given. Surely, it is of great interest to find the general coherent-mode structure of any arbitrary propagation-invariant field. At first sight this may be done by trying to solve the integral equation (3) with a kernel taken in the form of the exact solution for propagation-invariant field. However, this way leads to the unsolvable mathematical difficulties. In this paper, we try to find the coherent-mode structure of propagation-invariant field by solving the differential equation for propagation of the coherent mode under condition (4). When doing so, we use the paraxial approximation for field propagation, which is quite justifiable in the case of propagation-invariant fields.

3 General Solution

The propagation of the planar cross-spectral density function $W(\mathbf{x}_1,\mathbf{x}_2;z)$ in a free space, within the accuracy of the paraxial approximation, is described by the differential equation

$$\left(\nabla_{1\perp}^2-\nabla_{2\perp}^2+2ik\frac{\partial}{\partial z}\right)W(\mathbf{x}_1,\mathbf{x}_2;z)=0, \tag{5}$$

where $\nabla_{1\perp}^2$ and $\nabla_{2\perp}^2$ are the transverse Laplacian operators acting on coordinates of $\mathbf{x}_1$ and $\mathbf{x}_2$, respectively, and k is the wave number. Since the cross-spectral densities of the coherent modes of the field obey the same propagation equation, it follows that

$$\left(\nabla_{1\perp}^2-\nabla_{2\perp}^2+2ik\frac{\partial}{\partial z}\right)\Phi_n^*(\mathbf{x}_1;z)\Phi_n(\mathbf{x}_2;z)=0. \tag{6}$$

Expressing the function $\Phi_n(\mathbf{x};z)$ through its spectrum $\widetilde{\Phi}_n(\mathbf{u};z)$,

$$\Phi_n(\mathbf{x};z)=\int_{-\infty}^{\infty}\widetilde{\Phi}(\mathbf{u};z)\exp(i2\pi\mathbf{u}\cdot\mathbf{x})d\mathbf{u} \tag{7}$$

and interchanging the order of differentiation and integration, one may rewrite Eq. (6) as

$$\iint_{-\infty}^{\infty}\left(\nabla_{1\perp}^2-\nabla_{2\perp}^2+2ik\frac{\partial}{\partial z}\right)\widetilde{\Phi}_n^*(\mathbf{u}_1;z)\Phi_n(\mathbf{u}_2;z) \times\exp\left[i2\pi\left(\mathbf{u}_2\cdot\mathbf{x}_2-\mathbf{u}_1\cdot\mathbf{x}_1\right)\right]d\mathbf{u}_1d\mathbf{u}_2=0. \tag{8}$$

After straightforward application of operators $\nabla_{1\perp}^2$ and $\nabla_{2\perp}^2$ in Eq. (8) we obtain

$$\iint_{-\infty}^{\infty}\left[\frac{\partial}{\partial z}+i\frac{2\pi^2}{k}\left(\mathbf{u}_1^2-\mathbf{u}_2^2\right)\right]\widetilde{\Phi}_n^*(\mathbf{u}_1;z)\Phi_n(\mathbf{u}_2;z) \times\exp\left[i2\pi\left(\mathbf{u}_2\cdot\mathbf{x}_2-\mathbf{u}_1\cdot\mathbf{x}_1\right)\right]d\mathbf{u}_1d\mathbf{u}_2=0, \tag{9}$$

whence it follows that

$$\left[\frac{\partial}{\partial z}+i\frac{2\pi^2}{k}\left(\mathbf{u}_1^2-\mathbf{u}_2^2\right)\right]\widetilde{\Phi}_n^*(\mathbf{u}_1;z)\widetilde{\Phi}_n(\mathbf{u}_2;z)=0. \tag{10}$$

As is well known, the general solution of Eq. (10) is

$$\widetilde{\Phi}_n^*(\mathbf{u}_1;z)\widetilde{\Phi}_n(\mathbf{u}_2;z)=\widetilde{\Phi}_n^*(\mathbf{u}_1;0)\widetilde{\Phi}_n(\mathbf{u}_2;0) \times\exp\left[-i\frac{2\pi^2}{k}z\left(\mathbf{u}_1^2-\mathbf{u}_2^2\right)\right]. \tag{11}$$

A 4-D Fourier transform of both sides of Eq. (11) with respect to the polar coordinates (r,φ) used for spatial frequencies gives

$$\begin{aligned}\Phi_n^*(\mathbf{x}_1;z)\Phi_n(\mathbf{x}_2;z)=&\iint_0^{\infty}\iint_0^{2\pi}\widetilde{\Phi}_n^*(r_1,\varphi_1;0)\\ &\times\widetilde{\Phi}_n(r_2,\varphi_2;0)\exp\left[-i\frac{2\pi^2}{k}z\left(r_1^2-r_2^2\right)\right]\\ &\times\exp\left[i2\pi\left(x_2r_2\cos\varphi_2+y_2r_2\operatorname{sen}\varphi_2\right)\right]\\ &\times\exp\left[-i2\pi\left(x_1r_1\cos\varphi_1+y_1r_1\operatorname{sen}\varphi_1\right)\right]\\ &\times r_1r_2d\varphi_1d\varphi_2dr_1dr_2.\end{aligned} \tag{12}$$

Let us assume now that the cross-spectral density meets the propagation-invariant condition (4). Obviously, in this case,

$$\Phi_n^*(\mathbf{x}_1;z)\Phi_n(\mathbf{x}_2;z)=\Phi_n^*(\mathbf{x}_1;0)\Phi_n(\mathbf{x}_2;0). \tag{13}$$

Applying to Eq. (12), we come to the conclusion that the equality (13) may be fulfilled when and only when

$$\widetilde{\Phi}_n(r,\varphi;0) = Q_n(r,\varphi)\delta(r - r_{0n}), \tag{14}$$

where $Q_n(r,\varphi)$ is an arbitrary complex function, $\delta(r)$ is the Dirac function and r_{0n} is a parameter. Really on substituting for $\widetilde{\Phi}_n(r,\varphi;0)$ from Eq. (14) into Eq. (12) and making use of the sifting property of the Dirac function, we obtain

$$\begin{aligned}\Phi_n^*(\mathbf{x}_1;z)\Phi_n(\mathbf{x}_2;z) = r_{0n}^2 \int\int_0^{2\pi} Q_n^*(r_{0n},\varphi_1) \\ \times Q_n(r_{0n},\varphi_2)\exp[i2\pi r_{0n}(x_2\cos\varphi_2 + y_2 \operatorname{sen}\varphi_2)] \\ \times \exp[-i2\pi r_{0n}(x_1\cos\varphi_1 + y_1 \operatorname{sen}\varphi_1)]\,d\varphi_1 d\varphi_2, \end{aligned} \tag{15}$$

whence the equality (13) follows immediately. Using the polar coordinates (ρ,θ) in the **x**-plane, we find from Eq. (15) that

$$\begin{aligned}\Phi_n(\rho,\theta;z) = r_{0n}\int_0^{2\pi} Q_n(r_{0n},\varphi) \\ \times \exp[i2\pi r_{0n}\rho\cos(\varphi-\theta)]\,d\varphi.\end{aligned} \tag{16}$$

Then, expanding $Q_n(r_{0n},\varphi)$ into Fourier series,

$$Q_n(r_{0n},\varphi) = \sum_{p=-\infty}^{\infty} q_{np}\exp(ip\varphi), \tag{17}$$

and recalling the integral representation of the Bessel function, we may rewrite Eq. (16) in the following form:

$$\Phi_n(\rho,\theta;z) = 2\pi r_{0n}\sum_{p=-\infty}^{\infty} i^p q_{np}\exp(ip\theta)J_p(2\pi r_{0n}\rho) \tag{18}$$

where J_p denotes the Bessel function of the first kind and of order p.

We will recall now that, to describe the coherent-mode structure of the field, functions $\Phi_n(\rho,\theta;z)$ must be orthonormal, i.e.,

$$\int_0^R\int_0^{2\pi}\Phi_n^*(\rho,\theta;z)\Phi_m(\rho,\theta;z)\rho\,d\rho\,d\theta = \delta_{nm}, \tag{19}$$

where δ_{nm} is the Kronecker symbol and the radial integration is performed within the finite domain S of radius R. On substituting for Φ_n and Φ_m from eq. (18) into Eq. (19) and realizing the azimuthal integration with due regard for

$$\int_0^{2\pi}\exp[i(s-p)\theta]\,d\theta = 2\pi\delta_{ps}, \tag{20}$$

we obtain

$$\begin{aligned}8\pi^3 r_{0n}r_{0m}\sum_{p=-\infty}^{\infty} q_{np}^* q_{mp}\int_0^R J_p(2\pi r_{0n}\rho) \\ \times J_p(2\pi r_{0m}\rho)\rho\,d\rho = \delta_{nm}.\end{aligned} \tag{21}$$

In that way, we have shown that functions $\widetilde{\Phi}_n(r,\varphi;0)$ given by Eq. (18) with the parameters q_{np} and r_{0n} determined by the orthonormality condition (21) describe the coherent-mode structure of any propagation-invariant field.

4 Partial Solutions

The partial solutions for coherent modes of propagation-invariant field can be obtained by different choice of the parameters q_{np} and r_{0n} as they satisfy the condition (21).

The first choice is

$$\begin{aligned} q_{np} &= \begin{cases} q_{0p} & \text{for } n = 0, \\ 0 & \text{for } n \neq 0, \end{cases} \\ r_{0n} &= r_0, \end{aligned} \tag{22}$$

where q_{0p} and r_0 may be taken arbitrary. In this case the coherent-mode structure of propagation-invariant field is described by the only function

$$\Phi_0(\rho,\theta;z) = 2\pi r_0\sum_{p=-\infty}^{\infty} i^p q_{0p}\exp(ip\theta)J_p(2\pi r_0\rho), \tag{23}$$

that is the well known general solution for nondiffracting beams [4]. In the particular case, when

$$q_{0p} = \begin{cases} q_{00} & \text{for } p = 0, \\ 0 & \text{for } p \neq 0, \end{cases} \tag{24}$$

this solution takes the form

$$\Phi_0(\rho,\theta) = 2\pi r_0 q_{00} J_0(2\pi r_0\rho), \tag{25}$$

known as the fundamental Bessel beam [5].

Now we choose

$$q_{np} = \begin{cases} q_{nn} & \text{for} \quad p = n, \\ q_{nn}^* & \text{for} \quad p = -n, \\ 0 & \text{for} \quad p \neq \pm n, \end{cases} \tag{26}$$

allowing the parameter r_{0n} to take arbitrary values. It is a straightforward matter to make sure that this choice, with the value of q_{nn} taken so that

$$|q_{nn}|^2 = \left[16\pi^3 r_{0n}^2 \int_0^R J_n^2(2\pi r_{0n}\rho)\rho d\rho\right]^{-1}, \tag{27}$$

satisfies orthonormality condition (21). Then, substituting for q_{np} from the relation (26) into Eq. (18), we find that the coherent-mode structure of propagation-invariant field, this time, is described by the set of functions

$$\Phi_n(\rho,\theta;z) = 2\pi r_{0n}\left[i^n q_{nn}\exp(in\theta)J_n(2\pi r_{0n}\rho) + i^{-n}q_{nn}^*\exp(-in\theta)J_{-n}(2\pi r_{0n}\rho)\right]. \tag{28}$$

Substituting from Eq. (28) into Eq. (2) with due regard for a twofold degeneracy of the corresponding eigenfunction of the Fredholm integral equation (3) (except for $n = 0$), we find that the cross-spectral density function of propagation-invariant field takes the form

$$\begin{aligned} W(\rho_1,\theta_1,\rho_2,\theta_2;z) &= \lambda_0 4\pi^2 r_{00}^2 |q_{00}|^2 \\ &\times J_0(2\pi r_{00}\rho_1)J_0(2\pi r_{00}\rho_2) \\ &+ 2\sum_{n=1}^{\infty}\lambda_n 4\pi^2 r_{0n}^2 |q_{nn}|^2 \cos[n(\theta_1-\theta_2)] \\ &\times J_n(2\pi r_{0n}\rho_1)J_n(2\pi r_{0n}\rho_2). \end{aligned} \tag{29}$$

An interesting particular case of Eq. (29) may be obtained if we put $r_{0n} = r_0$ and choose

$$\lambda_n = \left(4\pi^2 r_0^2 |q_{nn}|^2\right)^{-1}, \tag{30}$$

In this case, in accordance with the summation theorem of Bessel function, we obtain

$$W(\rho_1,\theta_1,\rho_2,\theta_2;z) = J_0\left\{2\pi r_0\left[\rho_1^2+\rho_2^2-2\rho_1\rho_2\cos(\theta_1-\theta_2)\right]^{1/2}\right\}. \tag{31}$$

The field with the cross-spectral density (31) is known as the Bessel-correlated field with the uniform intensity distribution within the finite domain of radius R [6].

Our third choice for the parameters q_{np} and r_{0n} is

$$q_{np} = \begin{cases} q_{n\mu} & \text{for} \quad p = \mu, \\ q_{n\mu}^* & \text{for} \quad p = -\mu, \\ 0 & \text{for} \quad p \neq \pm \mu, \end{cases} \tag{32}$$

$$r_{0n} = \frac{\alpha_{\mu,n+1}}{2\pi R},$$

where μ is any positive integer and $\alpha_{\mu,n+1}$ is the $(n+1)$th zero of the Bessel function J_μ . Using the orthonormality relation for the Bessel function,

$$\int_0^R J_\mu\left(\alpha_{\mu,n+1}\frac{\rho}{R}\right)J_\mu\left(\alpha_{\mu,m+1}\frac{\rho}{R}\right)\rho d\rho = \delta_{nm}\frac{R^2}{2}J_{\mu+1}^2(\alpha_{\mu,n+1}), \tag{33}$$

and taking $q_{n\mu}$ so that

$$|q_{n\mu}|^2 = \left[\pi\alpha_{\mu,n+1}^2 J_{\mu+1}^2(\alpha_{\mu,n+1})\right]^{-1}, \tag{34}$$

one may readily verify that the choice (32) also satisfies the orthonormality condition (21). Then, substituting for q_{np} and r_{0n} from the relations (32) into Eq. (18), we find that the coherent-mode structure of propagation-invariant field is described by the set of functions

$$\Phi_n^{(\mu)}(\rho,\theta;z) = \frac{\alpha_{\mu,n+1}}{R}\left[i^\mu q_{n\mu}\exp(i\mu\theta)J_\mu\left(\alpha_{\mu,n+1}\frac{\rho}{R}\right) + i^{-\mu}q_{n\mu}^*\exp(-i\mu\theta)J_{-\mu}\left(\alpha_{\mu,n+1}\frac{\rho}{R}\right)\right]. \tag{35}$$

Substituting from Eq. (35) into Eq.(2) (this time, the eigenfunction have twofold degeneracy for all n), we find that the cross-spectral density function of the propagation-invariant field takes the form

$$W^{(\mu)}(\rho_1,\theta_1,\rho_2,\theta_2;z) = \cos[\mu(\theta_1-\theta_2)]\sum_{n=0}^{\infty}\lambda_n^{(\mu)} \times \frac{\alpha_{\mu,n+1}^2}{R^2}|q_{n\mu}|^2 J_\mu\left(\alpha_{\mu,n+1}\frac{\rho_1}{R}\right)J_\mu\left(\alpha_{\mu,n+1}\frac{\rho_2}{R}\right). \tag{36}$$

The interesting examples of such propagation-invariant field may be given by choosing

$$\lambda_n^{(\mu)} = \left(\frac{\alpha_{\mu,n+1}^2}{R^2}|q_{n\mu}|^2\right)^{-1}. \tag{37}$$

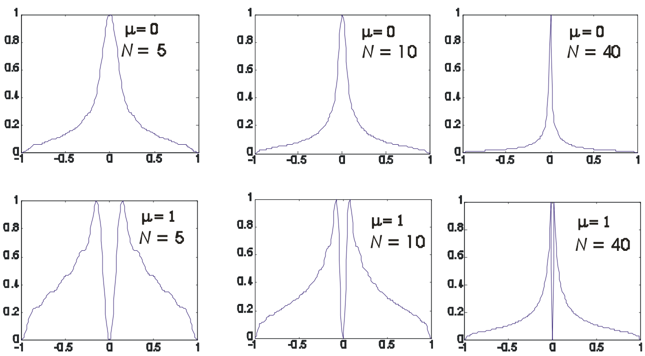
In this case, Eq. (36) takes the form

$$W^{(\mu)}(\rho_1,\theta_1,\rho_2,\theta_2;z)=\cos[\mu(\theta_1-\theta_2)]$$
$$\times\sum_{n=0}^{\infty} J_\mu\left(\alpha_{\mu,n+1}\frac{\rho_1}{R}\right)J_\mu\left(\alpha_{\mu,n+1}\frac{\rho_2}{R}\right), \qquad (38)$$

and the corresponding intensity distribution of the field is

$$I^{(\mu)}(\rho,\theta)=W^{(\mu)}(\rho,\theta,\rho,\theta;z)=\sum_{n=0}^{\infty} J_\mu^2\left(\alpha_{\mu,n+1}\frac{\rho}{R}\right). \quad (39)$$

We calculated the intensity distribution (39) for the first two orders μ, truncating the summation with respect to index n by different values of the number N of zeros of the Bessel function. The results of this calculation are shown in Fig. 1. As one may conclude from this figure, when N tends to infinity, the propagation-invariant field with the intensity distribution $I^{(0)}$ represents the infinitely thin light beam, and the propagation-invariant field with the intensity distribution $I^{(1)}$ represents the infinitely thin light tube. In this sense, we term these fields as light string beam and light capillary beam, respectively.



Relative transverse coordinate, *x/R*

Fig. 1 Normalized cross-sections calculated in accordance with Eq. (39) for different values μ and different values of the truncating parameter of summation *N*.

5 Conclusions

The general coherent-mode structure of propagation-invariant field has been found as general solution of the differential equation for propagation of the coherent modes of the field. It has been shown that there exist three different partial solutions for coherent modes which define fundamentally different classes of propagation-invariant fields. While the examples of propagation-invariant fields of the first two classes are well known, the third class is represented by optical beams of a new type. Within this class we predicted the existence of so-called light string beam and light capillary beam. These beams are characterized by extremely sharply localized energy distribution in their transverse sections, that may be advantageously utilized in various communication and measurement problems.

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