Scaling in the Fraunhofer Region from Gratings with Complex Structure

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Abstract: - We develop the mathematical foundation for the construction of regular fractal functions with periodic envelope, using analog periodic functions as basic components. This method represents an extension of the results obtained for the case of binary functions, studied in some previous works. The representation of fractal sets through a product superposition of scaled periodic components can be related with the IFS (Iterated Function System) method. The study that we carried out use cosine functions that can be superimposed as a product to obtain the complex grating. We study the degree of scaling in the diffraction from such gratings.

Key-Words: - Fraunhofer region, diffraction, periodic grating, scaling, complex geometry, fractal

1 Introduction

During the last decade, scientists working in the more diverse areas have recognized that most of their experiments possess a special geometric complexity [1]. In a previous work, it has been demonstrated that some fractal structures can be obtained starting from binary periodic distributions [2, 3].

In this paper, we want to show that different complex forms can also be constructed by means of continuous functions.

This fact is important for applications in the processing of optical signals where some type of geometry can be required in the final intensity distribution.

It is also our interest to be able to develop the initial mathematical foundations for the construction of complex regular objects that can be used as diffraction gratings, and to study the properties of the scattered electromagnetic fields from these structures [4-7]. In this way, analog and digital fractal signals can be represented through periodic functions [8].

The method used here to represent fractal sets through the product superposition of periodic components, can be related with the Iterated Function Systems (IFS) method [9]. This means that, using a sequence of sets as union of disjoined intervals, the necessary intersection operations among the domains distributed in a periodically way can be carried out.

In this paper we build gratings with complex geometry, which can be fractal, and we demonstrate that the scaling factor included in such structures is directly included in the diffracted field at the Fraunhofer region. Some of such structures can be fractals and also we use real scaling factors.

2 Mathematic Foundations

A general expression for the product superposition between periodic scaled functions is [10, 11]:

$$F(x) = \prod_{k=0}^{N} C\left[A\frac{x - x_{0i}}{\Delta_k}\right] \qquad \Delta_k = \frac{L}{s^k} \tag{1}$$

A very simple class of periodic functions which can be used in the present study are, for example, the trigonometric functions. Then, we study the diffraction from gratings defined as $C=cos^2$ or $C=sin^2$ in Eq. (1). These type of gratings are plotted in Fig. 1, where the periodic components and the product in each step are shown.

2.1 Far-field Diffraction

The electromagnetic field $U(x_1,y_1,z)$ at the Fraunhofer region, for the superposition of two transmitances with a rotation between them, can be obtained from the general expression:

$$U(x_1, y_1, z) \propto \Im\{U(x_0, y_0, 0)\} \qquad (2)$$

being (x_0, y_0, z) the system of coordinates on the transmittance and (x_1, y_1, z) the system of coordinates on a transversal plane to the propagation along z. The operator $\Im\{\ \}$ indicates the Fourier transform in the space of frequencies.

3 Self-similarity Analysis

Now, we develop a theoretical study for the intensity distribution at the far field, considering cases in which the scaling factor s can take a real value, generalizing the scaling concept for non integer values.

A method for measuring the characteristic scaling factor of fractal structures was introduced by using the self-similarity function, defined as the correlation of this structure and a magnified version of itself. Although this parameter was applied initially to study the diffraction from Cantor gratings, later studies were carried out for other situations. It has been observed that the self-similarity function has a period that coincides with powers of s, although the values of the peaks for such function depend on the order or state of the fractal object. Here, fractal distribution of intensity, obtained from the diffraction of complex structures built using Eq. 1, with order 4, 5 and 6, have been used.

The self-similarity function is defined as:

$$S(m) = \frac{\int_R \left(I(x) - \langle I(x)\rangle\right)\left(I\left(\frac{x}{m}\right) - \left\langle I\left(\frac{x}{m}\right)\right\rangle\right) dx}{\sqrt{\int_R \left(I(x) - \langle I(x)\rangle\right)^2 \, dx \int_R \left(I\left(\frac{x}{m}\right) - \left\langle I\left(\frac{x}{m}\right)\right\rangle\right)^2 \, dx}} \qquad (3)$$

being R the region where this parameter is calculated and m is a independent variable which scale the intensity distribution.

The results obtained for some values of s real-valued are shown in Fig. 2, where the correlation peaks correspond to the power of s. We can observe that, in accordance with the scaling factor, the contrast varies with the increases of the number of periodic components included in the product. This is related with the form in that the crests and the valleys of the function components are distributed into the product. All the cases are for $C=\cos^2$ and $C=sin^2$ with a scaling factor $s=2$, 3, 3.5 and 4 respectively. In this way, we have demonstrated that the scaling factor is included in the diffraction patterns obtained with such transmittance.

4 Conclusion

A method to obtain continuous prefractal functions, by means of a simple method, is implemented. We used the well-known results for fractal binary functions, for which the construction is made through periodic functions (with values 0 and 1). Cosine and sine functions are used here, to obtain gratings with complex structure. Then, we conclude that the results for discrete signals can be extrapolated for continuous functions and the diffracted intensity can be characterized through the self-similarity function.

Another interesting result is the possibility to use scaling factors with real values, which are clearly reflected in the self-similarity function for the diffracted intensity.

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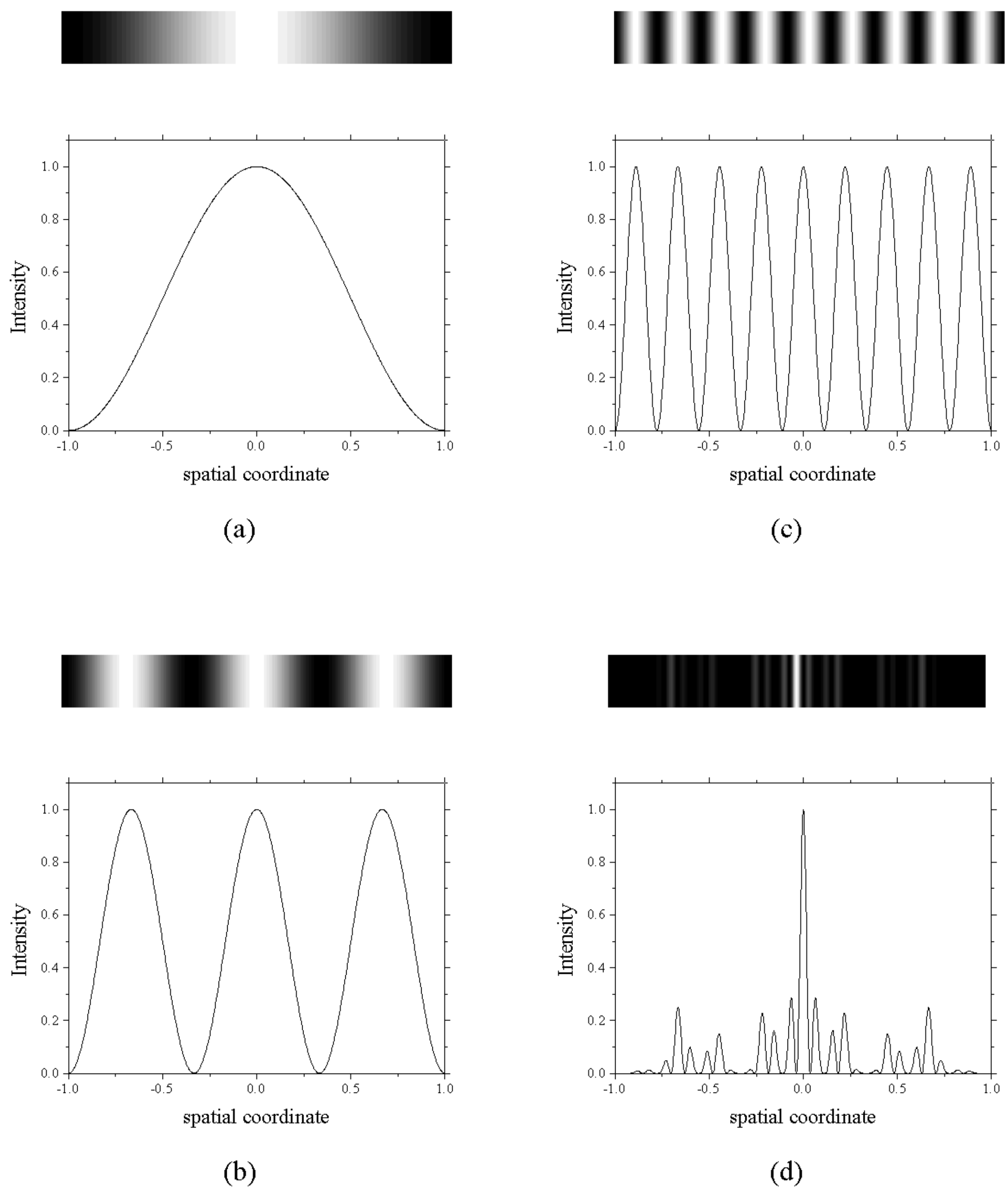

Fig. 1 – An example to obtain complex grating using periodic continuous functions $C=cos^2$, with an scaling factor $s=3$.

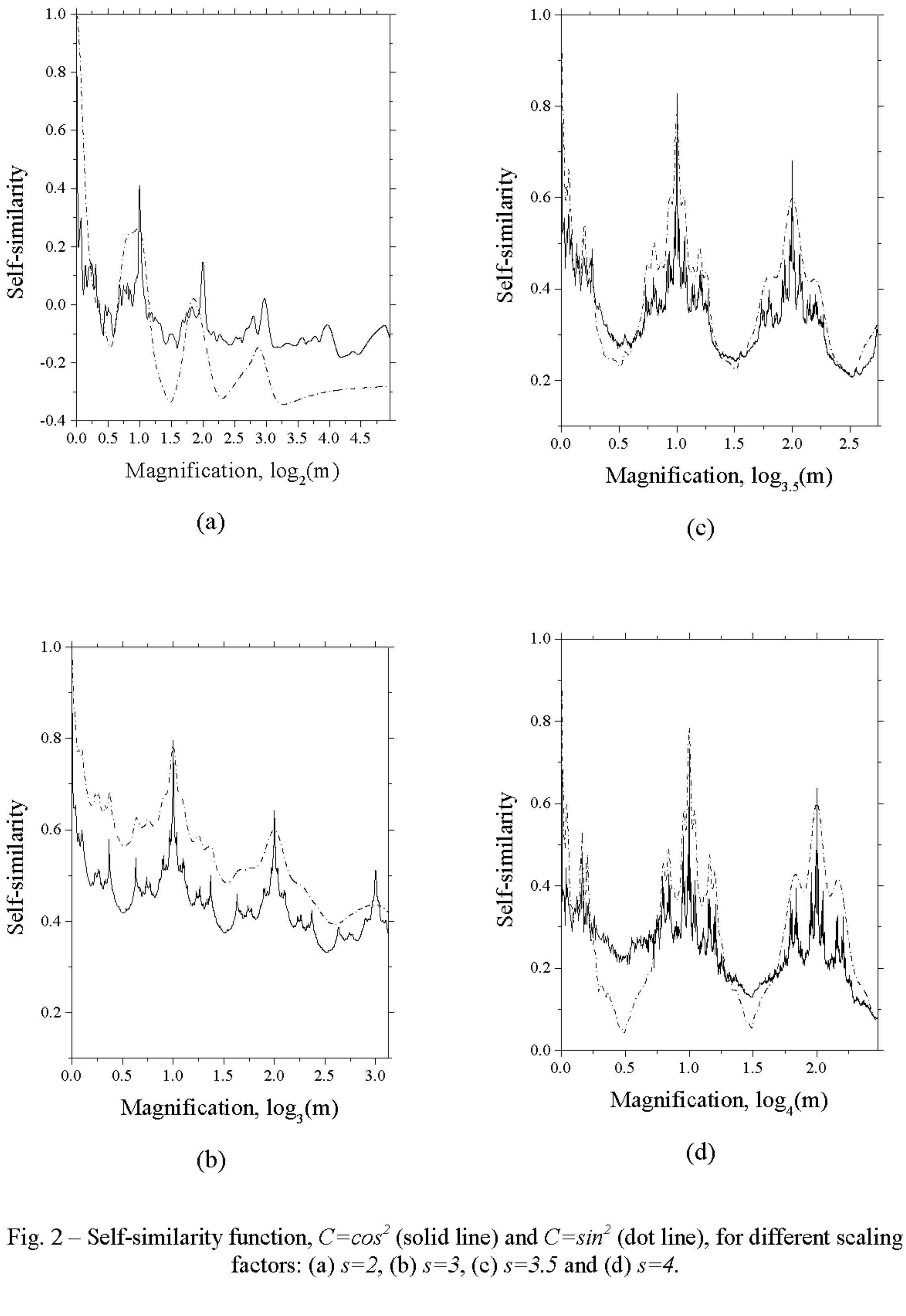

Fig. 2 – Self-similarity function, $C=cos^2$ (solid line) and $C=sin^2$ (dot line), for different scaling factors: (a) $s=2$, (b) $s=3$, (c) $s=3.5$ and (d) $s=4$.