Directional Self-similarity in the Superposition of Cantor Transmittances

DIANA CALVA MENDEZ[1], MARIO LEHMAN[2]

[1] Software Integral para Laboratorio (Sofilab) SACV
Lisboa 14-A, Colonia Juárez, Delegación Cuauhtémoc, México DF
MEXICO
dcalva@ienlaces.com.mx

[2] Centro de Investigaciones Opticas (CIC-CONICET), La Plata
ARGENTINA
mlehman@inaoep.mx

Abstract: - The properties of the diffracted field, when two Cantor diffraction gratings are superimposed are important in order to establish the relationships between the geometry of each fractal grating and the corresponding structure of the diffracted field. At the present paper, the case for studying is the characterization of field in the Fraunhofer region, into each envelope of diffraction, using the self-similarity function. For such problem it is shown that it is more appropriate the calculation on the modulus of the electromagnetic field, instead of using the intensity distribution. To exemplify, we use Cantor gratings with fractal dimensions *D=ln2/ln3* and *D=0.5*.

Key-Words: - Diffraction, Fraunhofer region, self-similarity, Cantor bars, fractal, gratings superposition

1 Introduction

The study of diffraction by using optical gratings and the applications of optics methods for image processing are fundamental areas of science and technology. For different applications, as those of our interest, diffraction gratings have been used with periodic and quasi-periodic structures [1,2]. However, there are few works where different geometries (including periodic and fractal structures) and their interaction with electromagnetic waves are studied [3-6]. In these cases, the point of view for such studies is in a general sense, and the periodic and quasi-periodic cases are specific examples.

A very valuable result from the point of view of the optical theory has been the introduction of the self-similarity function [7] which allows to evaluate the correlation degree (or fractality) between the intensity distribution of the electromagnetic field with a magnified version of itself.

From the point of view of the classical optics, there are many interesting effects that involve the superposition of diffraction gratings. These results can also be extended to the case of random objects. Furthermore, the basic properties can be applied for information processing by using optics methods. The generalization of properties toward new geometries would allow the extension of these methods and possibilities for the characterization of objects through the processing of optics signals [8, 9].

Here, we implement a simple mathematical formalism for the study of the self-similarity function when two Cantor fractal gratings, with a rotation between them, are superimposed. By using computational methods, we generate images of diffraction patterns for fractal Cantor gratings. With such results, we can study the variations in the self-similarity function, according with the directions on the plane that contain both diffraction gratings. This permits us to introduce the self-similarity tensor, that can be easily related with the directional properties on a surface.

2 Mathematical Basis

The method using periodic components to build fractal Cantor sets already has been used in previous papers [10, 11]. Beginning from a sequence of sets A^k periodically distributed and included into the initial domain A^0:

$$A_o \supset A_1 \supset A_2 \supset \dots ,$$
$$A_l = \bigcup_{i=1}^{M} H_{l,i} \qquad \text{with } H_{l,k} \cap H_{l,n} = \phi \tag{1}$$

where $H_{l,i}$ denotes contractions on the initial set A^0. If we use the characteristic function, defined as:

$$\chi[A] = \begin{cases} 1 & \text{si } x \in A \\ 0 & \text{si } x \notin A \end{cases} \quad (2)$$

it is very simple demonstrate that:

$$\chi\left[\bigcap_{l=0}^{N} A_l\right] = \prod_{l=0}^{N} \chi[A_l] \quad (3)$$

Then, for a certain N, the resulting invariant set is obtained with the operation:

$$F = \bigcap_{l=0}^{N} A_l \quad (4)$$

which represents a prefractal set. In the mathematical strict sense $N \to \infty$ is the fractal set. Two applications of this method is illustrated in Fig. 1.

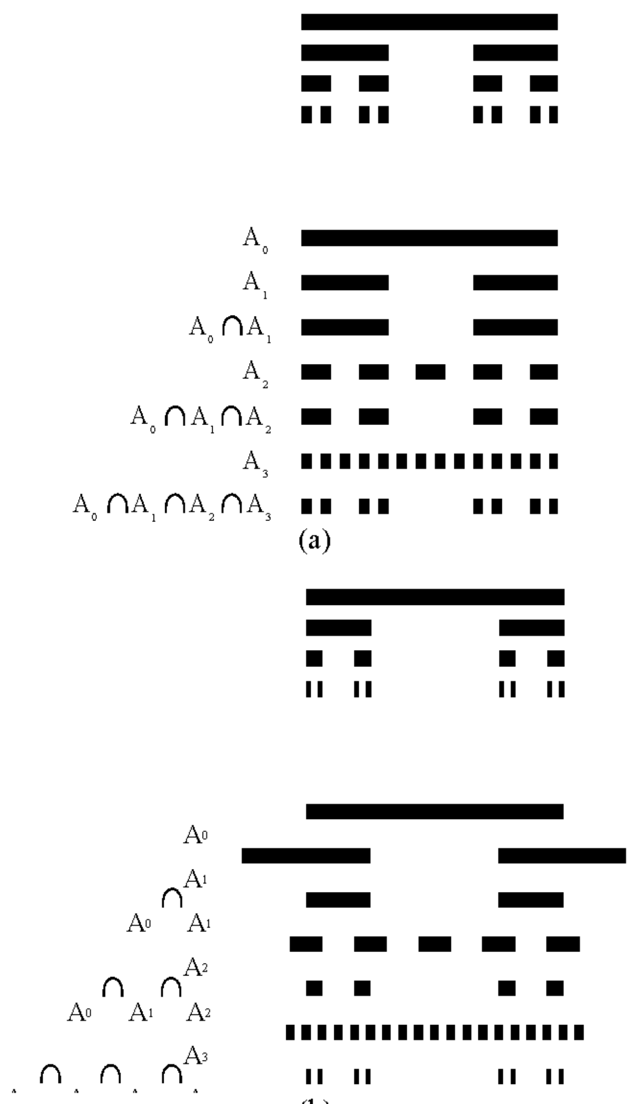


Fig. 1 – Cantor bars for different dimension: (a) $D=ln2/ln3$ and (b) $D=0.5$.

2.1 Superposition of Cantor sets

In this section, the mathematical expression for the superposition of two Cantor fractals when an angular displacement between them is taken into account is obtained, around a point of the x-y plane, where they are contained. An important theorem about the intersection between two fractal sets can be seen in Ref. [12], that will be useful for the construction that we want to make.

When there are two fractal sets with an angular displacement between them, denoted through a rigid movement σ, the characteristic function from Eq. (3) can be written, using rectangular functions:

$$\chi\left[A^n \cap \sigma\left(B^m\right)\right] = \left\{\sum_{n=0}^{M_1} rect\left[\frac{x_o - x_{on}}{\Delta_n}\right]\right\}^{g_k} \left\{\sum_{m=0}^{M_2} rect\left[\frac{x_0 + y_0 \cot\left(\frac{\pi}{2} - \theta\right) + x'_{0m}}{\Delta_m}\right]\right\}^{h_l} \quad (5)$$

where it has also been supposed, for further simplicity, that the structure represented through the set A^n preserve the symmetry along the y axis, such as can be seen in Fig. 2.

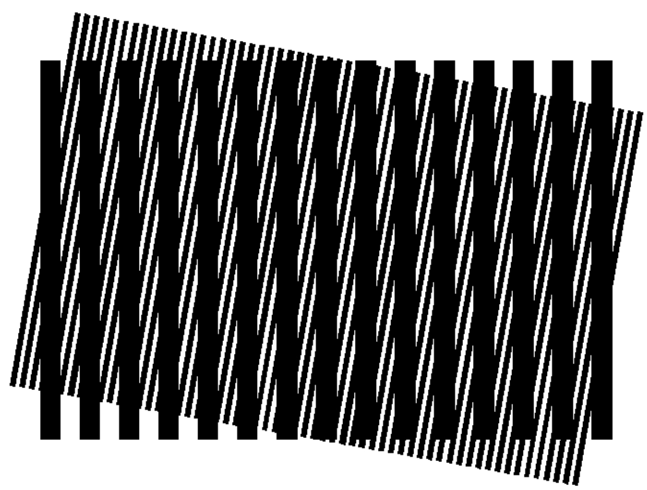


Fig. 2 – Superposition of two gratings with an angular difference between them.

For the expression in Eq. (5) is simple to show that the central points on the x-axis and the width of the intervals of each rectangular function before and later of the rotation, are related through [13]:

$$x'_{0m}(l) = \frac{x_{0m}(l)}{\cos(\theta)} \qquad \Delta'_m(l) = \frac{\Delta_m(l)}{\cos(\theta)} \quad (6)$$

where each singular domain have width Δ_m ($=L/3^{m-1}$) and the central points are given by:

$$x_{0m}(l) = -\frac{L}{2} + \frac{\Delta_m(l)}{2} + \Delta_m(l)(l-1) \qquad l = 1,\ldots,3^{l-1} \tag{7}$$

l and k are parameters indicating the order of the prefractal structure.

3 Field in the Fraunhofer Region

The electromagnetic field $U(x_1,y_1,z)$ at the Fraunhofer region, for the superposition of two transmitances with a rotation between them, can be obtained from the general expression:

$$U(x_1, y_1, z) \propto \Im\{U(x_0, y_0, 0)\} \tag{8}$$

being (x_0,y_0,z) the system of coordinates on the transmittance and (x_1,y_1,z) the system of coordinates on a transversal plane to the propagation along z. The operator $\Im\{\ \}$ indicates the Fourier transform in the space of frequencies.

If a density function is considered to represent two fractal transmittances, with an angular rotation θ between them, the electromagnetic field for the superposition of two Cantor gratings, can be expressed in a simple way through:

$$U(f_x, f_y, z) = \exp\left[i\left(kz + \frac{k}{2z}x_1^2\right)\right]\left(\frac{\Delta_1 \Delta_2}{\lambda z\, sin(\theta)}\right)$$
$$sinc\left[\pi f_x \frac{\Delta_2}{sin(\theta)}\right] sinc\left[\pi \Delta_1\left(f_x + f_y\, tan\left(\frac{\pi}{2}+\theta\right)\right)\right]$$
$$\sum_{j=1}^{N_2} C_2(x_{02j}) \exp\left[-2\pi i f_y \frac{x_{02j}}{sin(\theta)}\right] \tag{7}$$
$$\sum_{i=1}^{N_2} C_1(x_{01i}) \exp\left[-2\pi i x_{01i}\left(f_x + f_y\, tan\left(\frac{\pi}{2}+\theta\right)\right)\right]$$

where this expression is divided in two terms: a form factor (given by the functions sinc) and a structure factor (given by the double sum). The first term is related with the envelope of the diffraction pattern and the second with the fine structure, that is to say the form in which the smallest elements are distributed into the total structure that produce the diffraction.

3 Results Obtained

The degree of fractality in the electromagnetic field diffracted from Cantor fractal gratings can be quantified through the self-similarity function, which has already been defined as the correlation between the field intensity and a magnified version of itself [7]. It is calculated into a region R for the intensity distribution function, normalized with the envelope of this distribution. It has been observed that the self-similarity function has a certain periodicity which coincides with powers of the scaling factor of the Cantor fractal.

3.1 Self-similarity along a direction

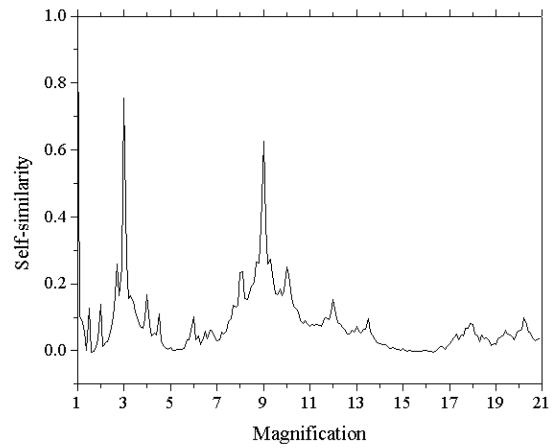
Now, we define the self-similarity function of the electromagnetic field is defined, along one coordinate, for the far field region. Here, the self-similarity function on the modulus of the electromagnetic field is calculated and not on the distribution of intensity, as originally defined.

The case of two-dimensional fractal gratings is considered with the self-similarity calculated for the x axis, which is defined as:

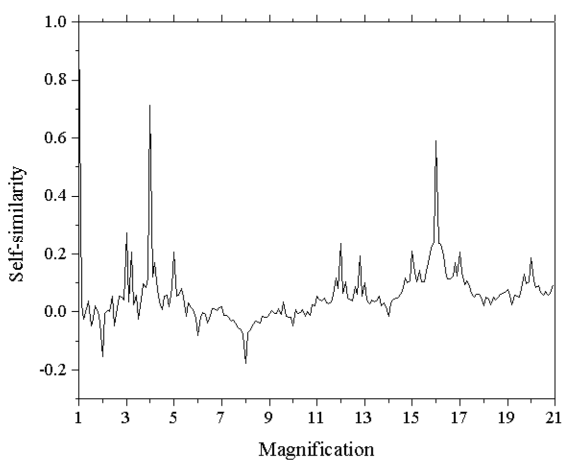
$$S(m) = \frac{\int_R \left(|U(\xi)| - \langle |U(\xi)| \rangle\right)\left(\left|U\left(\frac{\xi}{m}\right)\right| - \left\langle \left|U\left(\frac{\xi}{m}\right)\right| \right\rangle\right) d\xi}{\sqrt{\int_R \left(|U(\xi)| - \langle |U(\xi)| \rangle\right)^2 d\xi \int_R \left(\left|U\left(\frac{\xi}{m}\right)\right| - \left\langle \left|U\left(\frac{\xi}{m}\right)\right| \right\rangle\right)^2 d\xi}} \tag{8}$$

Single Cantor sets are used and then, because it is a two-dimensional grating which have variations along only one coordinate. The characterization through the self-similarity function can be observed at Fig. 3, for the superposition of Cantor gratings with scaling factor 3 and 4 respectively.

Fig. 4(a) shows the result that corresponds to the self-similarity calculated along a direction that has an angle of 45 degrees with the coordinate axes, but for two Cantor gratings with different dimension for an angle of 90 degrees between them. Each fractal grating have a scaling factor of 3 and 4 respectively and we can see the corresponding peaks present in the figures. Fig. 4(b) shows the results of self-similarity corresponding to an angle of 30 degrees between both fractal gratings, equally calculated along a direction of 45 degrees. In all cases it can be seen that the scaling factor of each grating are present, although with smaller contrast that for the main directions along the x and y axes.



(a)



(b)

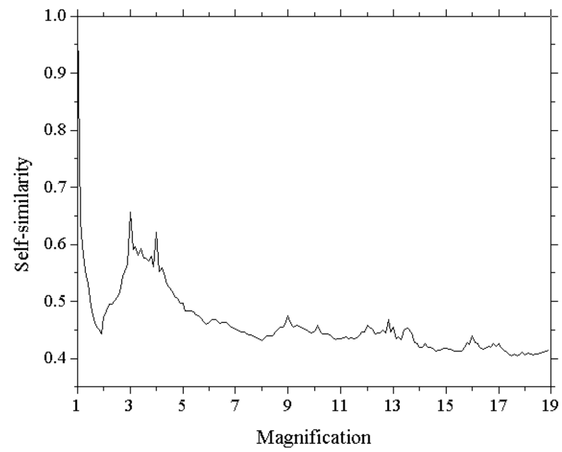
Fig. 3 – Self-similarity for different scaling factor in the grating: (a) $s=3$ and (b) $s=4$.

2.1 Self-similarity Tensor.

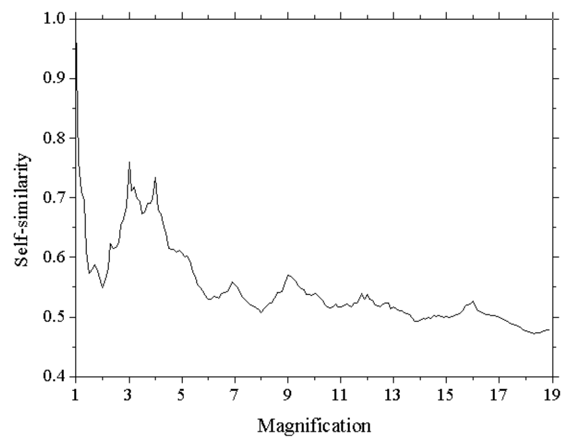
In many cases two or more effects can be combined, therefore it is useful to define a composed self-similarity function. That is to say, in some cases it is necessary to determine the variation along a specific direction, knowing the properties along principal directions. We introduce for this the self-similarity tensor given by:

$$S = \begin{Vmatrix} S_{xx} & S_{xy} \\ S_{yx} & S_{yy} \end{Vmatrix} \qquad (9)$$

which correlates the distributions of intensity for two positions along one or two principal axis.



(a)



(b)

Fig. 4 – Self-similarity along a direction for two different angles between both transmittances: (a) 90 degree and (b) 30 degree.

4 Conclusion

The study of diffraction by optical fractal gratings for applications to the optical information processing has been presented in this paper. We have seen some interesting properties of the self-similarity function introduced by T. Asakura et al. [7], related with the angular superposition of Cantor gratings. The subject of study of diffraction by fractal gratings can be easily extended to random and quasi-random structures.

Since the construction method of fractal structures is related with the corresponding structure of the diffracted field, an introduction involving the

construction of Cantor gratings from periodic components was carried out.
The characteristics of the self-similarity function when two Cantor gratings are superimposed with an angular displacement between them were presented. Gratings with equal and different fractal dimension were used, and the self-similarity function reveals us the characteristics of fractality in the diffracted field into the Fraunhofer region. The scaling factors for both gratings are present in the field, although the correlation with the main direction (along the direction of fractality of the grating) is smaller when the angle is near to 45 degrees. However, the correlation of the modulus of field with itself along different directions maintains the peaks corresponding to the scaling factors. This way, different class of correlation (or self-similarity functions) can be defined inside the envelope of the diffraction pattern to characterize the structure of the electromagnetic field. Finally, it is worthy to highlight that the self-similarity calculations are developed on the modulus of the electromagnetic field (or the square root in the intensity measured) because it shows a bigger contrast regarding the results using the distribution of intensity.

References:

[1] D. Maystre, *Selected Papers on Diffraction Gratings*, SPIE MS 83, 1992.
[2] M. Born, E. Wolf, *Principles of Optics*, Pergamon Press, Oxford, 1975.
[3] C. Allain, M. Cloitre: Optical diffraction on fractals, *Phys. Rev.* B 33(5), 1986, pp. 3566-3569.
[4] A. Lakhtakia, N. S. Holter, V. K. Varadan, V. V. Varadan: Self-similarity in diffraction by a self-similar fractal screen, *IEEE Trans. Ant. Prop.* 35, 1987, pp. 236-239.
[5] J. L. Pombo, M. Lehman, M. Garavaglia and D. L. Jaggard, Quasi-Self-Similar Cantor gratings and their diffraction, *Microwave and Optical Tech. Lett.* 20(6), 1999, pp. 393-397.
[6] M. Lehman, Superposition of Planar Cantor Gratings and Directional Self-similarity I: General Considerations, to be published in *Optik*.
[7] Y. Sakurada, J. Uozumi, T. Asakura: Scaling properties of the Fresnel diffraction field produced by one-dimensional regular fractals, *Pure Appl. Opt.* 3, 1994, pp. 371-380.
[8] T. Alieva, F. Agullo-López: Optical wave propagation of fractal fields, Optics Comm. 125 (1996) 267-274 1996.
[9] J. Ebbeni, Overview of coherent optics applications in metrology, in *Scattering and Diffraction*, H. A. Ferweda, ed., *Proc. of SPIE* 1029, 1988, pp. 2-17.
[10] D. L. Jaggard, T. Spielman, Triadic Cantor target diffraction, *Microwave and Opt. Tech. Lett.* 5(9), 1992, pp. 460-466.
[11] M. Lehman: Diffraction by a fractal transmittance obtained as superposition of periodical functions, *Fractals* 6(4), 1998, pp. 313-326.
[12] K. Falconer, *Fractal Geometry*, John \& Wiley Sons, Inc., Chichester 1990.
[13] M. Lehman, Fractal Diffraction Gratings build through Rectangular Domains, *Optics Communications* 195(1-4), 2001, pp. 11-26.