Finite time synchronization of Lorenz-based chaotic systems

Leonardo Acho Zuppa[1]
Centro de Investigación y Desarrollo de Tecnología Digital (CITEDI-IPN)
Tijuana B.C., México
Correspondence: CITEDI-IPN 2498 Roll Dr. #757, Otay Mesa, San Diego California, 92154-7275, USA

César Cruz Hérnandez[2]
Centro de Investigación Científica y de Educación Superior de Ensenada (CICESE)
Ensenada B.C., México

Ana Y. Aguilar Bustos
Centro de Investigación Científica y de Educación Superior de Ensenada (CICESE)
Ensenada B.C., México

Abstract: - The present paper is about the design of a finite time synchronization algorithm applied to the Lorenz chaotic system. Lyapunov theory is used to prove finite time convergence.

Key-Words: - Finite time convergence, Synchronization, Lorenz system.

1 Introduction

The design of systems with finite time convergence has been studied, among others, in [6], [8], and [1]. Lately the synchronization problem has been of great interest, see for example, [2], [3], [4], [5] and references there in; however, no one of them has considered the problem of finite time convergence. In the case of discrete systems, there is a result of finite time convergence (see [7]). The objective of this paper is to present a finite time synchronization algorithm for the Lorenz chaotic system (the continuous case). Lyapunov theory is used to prove finite time convergence. Simulation results are shown to support our main result.

2 Problem Formulation

The Lorenz system is given by

$$\begin{aligned} \dot{x} &= \sigma(y-x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz \end{aligned} \qquad (1)$$

where σ, r, and b are constant parameters. With $\sigma=16$, $r=45.6$ and $b=4$, the Lorenz system presents chaotic behavior.

Remark 1 For electronics implementation of the system (1), it is suggested to use a transformation of variables to avoid wide dynamics range (see [4]).

The system (1) is referred as the transmitter, where $x(t)$ is the transmitted signal used by the receiver system given by

$$\begin{aligned} \dot{x}_r &= \sigma(y_r - x_r) + k_1 \sigma sign(x - x_r) \\ \dot{y}_r &= rx - y_r - xz_r + sign(x - x_r) \\ \dot{z}_r &= xy_r - bz_r \end{aligned} \qquad (2)$$

where k_1 is a positive constant and *sign(.)* is the Sign function which is one if the argument is positive, minus one if the argument is negative, and zero if the argument is zero. The above dynamics system is a modification of the receiver system proposed in [4] to get finite time convergence. This receiver system only use the transmitted signal as its input.

Let us define the state errors between the transmitter system and the receiver system one as

$$e_1 = x - x_r, \quad e_2 = y - y_r, \quad e_3 = z - z_r \qquad (3)$$

Subtracting (2) from (1) and using (3) we have

$$\begin{aligned} \dot{e}_1 &= \sigma(e_2 - e_1) - k_1 \sigma \, sign\,(e_1) \\ \dot{e}_2 &= -e_2 - xe_3 - sign\,(e_1) \\ \dot{e}_3 &= xe_2 - be_3 \end{aligned} \qquad (4)$$

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The synchronization problem can be stated using the error dynamics (4) and it is equivalent to asymptotic stabilice the error dynamics. The finite time synchronization problem consists that for any initial conditions $e_1(0)$, $e_2(0)$ and $e_3(0)$, the solution of the system (4) (unique in forward time!) $e_1(t)$, $e_2(t)$ and $e_3(t)$, reach the origin (e_1=e_2=e_3=0) in finite time, i.e, there exists a settling time t_s such that

$$\lim_{t\to t_s}\|e(t)\| = 0,$$

and

$$e(t) = 0 \quad \forall t \geq t_s$$

where $e(t)=[e_1 \ \ e_2 \ \ e_3 \ ]$.

This settling time t_s is a function of the initial conditions (see [6]). In the paper cited in [8], in its theorem 2 says that the system converges in finite time if given a Lyapunov function V, its time derivative along of the trajectories of the system under study, is bounded by a negative number ($\dot{V} \leq -k,\ k>0$). Global finite time convergence is obtained if the Lyapunov function V is proper (see [8]).

To prove finite time convergence for the system (4), the next Lyapunov function (a proper one) is proposed

$$V = \frac{1}{2}(\frac{1}{\sigma}e_1^2 + e_2^2 + e_3^2) + \frac{1}{\sigma}|e_1|$$

where its time derivative along of the trajectories of the system (4) yields

$$\begin{aligned}\dot{V} &= \frac{1}{\sigma}e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3 + \frac{1}{\sigma}sign(e_1)\dot{e}_1 \\ &= -e_1^2 + e_1 e_2 - e_2^2 - be_3^2 - |e_1| - k_1 - k_1|e_1| \\ &= -(e_1 - \frac{1}{2}e_2)^2 - \frac{3}{4}e_2^2 - be_3^2 - |e_1| - k_1 - k_1|e_1| \\ &\leq -k_1\end{aligned}$$

Such that with $k_1>0$, the finite time convergence for the system (4) is obtained after theorem 2 in [8] is invoked.

To estimate the settling time, we can integrate the expression obtained above,

$$\dot{V} \leq -k_1. \qquad (5)$$

Integrating (5) from 0 to t_s, we obtain

$$-V(0) \leq V(t_s) - V(0) \leq -k_1 t_s$$

which means that

$$\begin{aligned}t_s &\leq \frac{V(0)}{k_1} \\ &= \frac{1}{k_1}[\frac{1}{2}(\frac{1}{\sigma}e_1^2(0) + e_2^2(0) + e_3^2(0)) + \frac{1}{\sigma}|e(0)|].\end{aligned}$$

3 Simulation Results

The systems (1) and (2) were programmed using MathLab. Fig. 1 shows the state variables of the transmitter and the receiver. The initial conditions used were $x(0)=5$, $y(0)=-10$, $z(0)=15$, and $x_r(0)=-5$, $y_r(0)=10$, and $z_r(0)=-15$. The value of k_1 used was one. Fig. 2 shows the error dynamics.

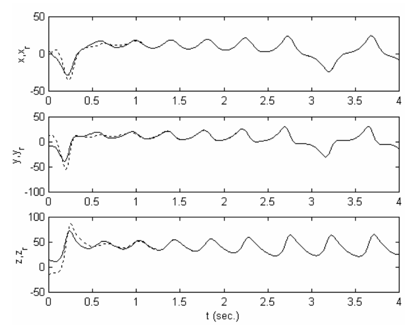

Fig.1 State variables of the transmitter (continuous lines) and the receiver (dotted lines): Top picture is x and x_r, middle picture is y and y_r, and bottom picture is z and z_r.

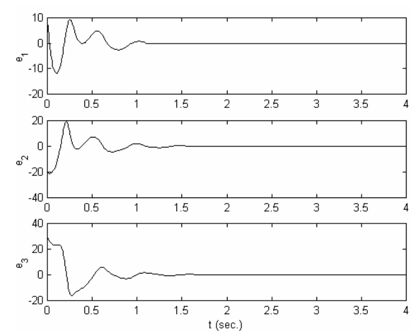

Fig. 2 State variables of the error dynamics: Top picture is e_1, middle picture is e_2, and bottom picture is e_3.

Remark 2 The synchronization algorithm presented here can be used to implement chaotic signal masking systems, as shown in [4].

4 Conclusion

In the present paper a finite time synchronization algorithm is presented using the Lorenz system. Lyapunov theory is used to prove finite time convergence. This synchronization algorithm can be used to design chaotic signal masking systems, as shown in [4]. Numerical results are shown to support our results.

References:

[1] Bhat S. P. and Bernstein D. S. Finite-Time Stability of Continuous Autonomous Systems, *Siam J. Control Optim.*, Vol.38 No.3, 2000, pp. 751-766.

[2] Chua Leon O., Yang Tao, Zhong Guo-Qun and Wu Chai Wah. Adaptive Synchronization of Chua's Oscillators, *International Journal of Bifurcation and Chaos*, Vol.6, No.1, 1996, pp. 189-201.

[3] Corron Ned J. and Hahs D. A New Approach to Communications Using Chaotic Signals, *IEEE Trans. on Circuits and Systems-I* , Vol.44, No.5, 1997, pp. 373-458.

[4] Cuomo Kevin M. and Oppenheim Alan V. Synchronization of Lorenz-based Chaotic Circuits with Application to Communications, *IEEE Trans. on Circuits and Systems-II*, Vol.40 No.10, 1993, pp. 626-633.

[5] Dedieu H., Kennedy P. M., and Hasler M. Chaos Shift Keying: Modulation and Demodulation of a Chaotic Carrier Using Self-Synchronization Chuas's, *IEEE Trans. on Circuits and Systems-II*, Vol.40 No.10, 1993, pp. 634-642.

[6] Hong Yigwruang, Huang J., and Xu Yangsheng. On an Output Feedback Finite-Time Stabilization Problem, *IEEE Trans. on Automatic Control*, Vol.46 No.2, 2001, pp. 305-309.

[7] Millerioux M. and Mira C. Finite-Time Global Synchronization for Piecewise Linear Maps, *IEEE Trans. Circcuits and Systems I*, Vol. 48, No.1, 2001, pp 111-116.

[8] Paden Brad E. and Sastry Shankar. A Calculus for Computing Filippov's Differential Inclusion with Application to the Variable Structure Control of Robot. *IEEE Trans. on Circuit and Systems*, Vol.Cas-35, No.1, 1987, pp. 73-82.