ALGORITHM FOR THE SOLUTION OF A SPECIAL TYPE OF MINIMAX PROBLEM IN INTEGERS

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Abstract:- An original algorithm is set forth in connection with a problem that arose with a large arrangement of electric devices. The problem was reduced to one of permutations of a given number of integers into equally sized groupings, in such a way that sums within those groupings had to be set as alike as possible. Since the problem was apparently new, there was scant literature on the subject. Fortunately enough, an ingenious solution was finally found, and it proved to be both easy to understand and easy to convert into almost any programming language.

Key –Words: c

1 Introduction

The problem dealt with in this paper arose with the need to balance a certain number of electric devices that had to be distributed on a 10 × 3 board, which was itself connected to a transformer whose capacity depended on the potential of the column that registered the highest voltage. Since the total electric charge to be put onto the board was fixed, the best option seemed to be an attempt to distribute the total charge as equally as possible along the three columns. Accordingly, this was tantamount to distributing a set of given integers into certain subsets or groupings of equal size, in such a way that the sums within the groupings of numbers were made as even as possible.

From the mathematical viewpoint, this is a problem of permutations, yet on the face of it the total number of permutations to deal with on the board was simply of an astronomic size, as it is readily seen that the factorial of 30 (30!=265252859812191058636308480000000) is well nigh impossible to handle. Therefore it became necessary to look for a more convenient and orderly method to tackle the problem, in a way that each new step should improve on the previous one, and by this we mean that sums of integers along the corresponding columns were set as close to the average as possible.

After looking over some literature on the subject ([1], [2], [3], [4], [5]), a few general methods suggested by different authors were finally discarded, as they all seemed to introduce too many unnecessary complications.

The algorithm suggested in this paper solves the problem by means of spanning a sequence of nested intervals which contains all the sums along columns and which works equally well for any number of integers or groupings

2 Problem Formulation

Let us assume that $k \times n$ integers are given and that they are to be distributed in a matrix of that order, in such a way that the greatest sum of elements along columns be as small as possible. Moreover, there is no loss of generality if we further assume that the number of integers to be distributed is a multiple of the number of columns, as if that should not happen to be the case, we could simply add up zeros till the nearest multiple of n is reached.

Let $P = \frac{1}{n}\sum_{j=1}^{n}\sum_{i=1}^{k} a_{ij}$ be the column average of all n integers to be distributed along the n columns. Then it is found necessary to take

$$\acute{min}\left(\underset{1\le j\le n}{\acute{max}}\left|\sum_{i=1}^{k} a_{ij} - P\right|\right)$$

Notice that, since all numbers are supposed to be integers, we cannot think of a continuous approach to our goal. Hence the optimal approach must be found by means of suitable permutations of the given integers. On the other hand, there is no starting point that might be considered at the outset as being “better” or “worse” by itself, inasmuch as a single change or a single permutation could give rise to a great difference among the sums of integers along columns. Therefore our search has to be such that a sort of “inverse entropy” is attained, i.e., in a way that the process should never pass from a “better” distribution to a “worse” one.

Accordingly, the algorithm we come up with is developed as follows: In the first place, we start out with an arbitrary distribution of the given integers in the matrix. Then we introduce the following process of orderly exchanges of elements among different columns:
Let j_M be the number of the column where the sum of its elements has the largest value and let us denote this value by S_{j_M}. Likewise, let j_m be the number of column where the sum of its elements is the least, and let us denote this sum by S_{j_m}. Now let us exchange some elements between the columns j_M y j_m with a view to reducing the size of the interval $[S_{j_m}, S_{j_M}]$ where all the column sums are found. Since $S_{j_m} \le P \le S_{j_M}$, then we must see to it that our algorithm may create a sequence of nested intervals which contain *P*.

Let us write

$$d_j = S_j - P \;, \qquad j = 1,2,\ldots n, \qquad (1)$$

which is the difference between the corresponding sum of the *j*-th column and the average.

Definitión: An exchange of elements between the two columns j_M y j_m is said to be ***permissible*** if as a result it follows that the value S_{j_m} increases by Δ, in such a way that $S_{j_m} < S_{j_m} + \Delta < S_{j_M}$, (and at the same time $S_{j_m} < S_{j_M} - \Delta < S_{j_M}$). Thus $0 < \Delta < S_{j_M} - S_{j_m}$, and, taking into account condition (1), we may write

$$0 < \Delta < d_{j_M} - d_{j_m} \qquad (2)$$

This condition is of utmost importance in the development of the algorithm, as it guarantees that the ***sequence of nested intervals*** in the matrix will yield a better distribution of the elements within the matrix in comparison with the preceding step

But now an important question arises, namely: out of all permissible exchanges, which particular one will prove to be the most suitable to choose? A good variation consists in carrying out the exchange corresponding to the particular Δ which is closest to $\Delta^* = \dfrac{d_{j_M} - d_{j_m}}{2}$, the latter being the amount that brings together the sums of elements that lie on the columns j_M and j_m.

Once an exchange of elements between columns has taken place, new values of j_m y j_M will be determined, so as to look for the possibility of another exchange. In case that no permissible exchange should happen to be found between the two columns j_m and j_M, then an exchange will be sought between the j_m-th column and that whose sum has an immediate lower value than S_{j_M}, and so forth. If no such possibilities were to be found with j_m, then the same procedure would be carried out with the j_M-th column as well as the remaining ones Thereafter, the same procedure will be repeated with the column whose sum of elements comes immediately lower than S_{j_M} or else with that column whose sum of elements is immediately higher than S_{j_m}. This procedure is repeated stepwise until all possibilities have been taken into account. Every time that an exchange of elements is performed, the new values of j_M and j_m will be set into their places, and the procedure will be repeated from the beginning. It will only stop when there is no possibility for any further permissible exchange.

3 Algorithm

Starting out from any arbitrary initial distribution, the proposed algorithm has the following steps:

1. The elements in each column are arranged in decreasing order, i.e:
   $$a_{i+1j} \le a_{ij}\,, \qquad 1 \le i \le k-1,\; 1 \le j \le n$$
2. The sums S_j are calculated, and thus the values of j_m and j_M are determined.

3. The quantities $d_{j_m}, d_{j_M}, \Delta^*$ are calculated.

4. A rectangular arrangement $B = \{b_{pq}\}_{kk}$ is set, where each $b_{pq} = a_{pj_M} - a_{qj_m} = \Delta$ represents the variation of the sums of elements in the columns j_M and j_m, owing to the exchange between a_{pj_M} and a_{qj_m}. By the same token $b* = b_{p*q*}$ must be selected in such a way that $0 \le b* \le \Delta$ is a permissible exchange and, $|b* - \Delta*| = mín|b_{pq} - \Delta^*|$. In this way, the best possible exchange between the elements in both columns j_m and j_M will be attained, namely the exchange of a_{p*j_M} and a_{q*j_m}

5. then we proceed to calculate the sums $b_{p_1q_1} + b_{p_2q_2}$ for $p_1, p_2, q_1, q_2 = 1,2,...k$, $p_1 < p_2$, $q_1 \ne q_2$ and choose

$b** = mín\left|\left(b_{p_1q_1} + b_{p_2q_2}\right) - \Delta*\right|$.

Notice further that each sum of the type

$b_{p_1q_1} + b_{p_2q_2} = (a_{p_1j_M} - a_{q_1j_m}) + (a_{p_2j_M} - a_{q_2j_m}) =$

$= (a_{p_1j_M} + a_{p_2j_M}) - (a_{q_1j_m} + a_{q_2j_m})$ represents the variation of S_{j_M} and S_{j_m} if pairs of elements are simultaneously exchanged, as there might happen that the exchanges corresponding to $b_{p_1q_1}$ and $b_{p_2q_2}$ might not be the best one, or event that those exchanges should not happen to be permissible. The latter case might happen, for instance, if one of them were too big and the other one would have a negative sign. Nevertheless the simultaneous exchange of $a_{p_1j_M}$ and $a_{p_2j_M}$ for $a_{q_1j_m}$ and $a_{q_2j_m}$ yields a result that is closest to Δ^*.

It is interesting to notice that in fact we need not carry out all of the sums $b_{p_1q_1} + b_{p_2q_2}$ as, on account of step one on table *B*, the elements will satisfy the following inequality:

$$b_{pq} \le b_{pq+1}, \quad b_{pq} \ge b_{p+1q} \qquad (3)$$

so that, starting with $p_1 = 1$, $q_1 = 1,2,...,k$, provided that $b_{p_1q_1}$ is fixed, the sum $b_{p_1q_1} + b_{p_2q_2}$ (with $p_1 < p_2$, $q_1 \ne q_2$) becomes greater than Δ^*, the next sum to be calculated is $b_{p_1q_1} + b_{p_{2+1}q_2}$, but if this should happen to be less Δ^*, then the following one is $b_{p_1q_1} + b_{p_2q_{2+1}}$. In this way, the advance in table B takes place stepwise, and the procedure will stop at the time when, having reached the *k*-th column we must move downwards

6. Next we compare $b*$ and b^{**}, and thus choose the one which is closest to Δ* , whereupon the best exchange is accomplished..

7. Return to step one.

4 Example

Let us illustrate the algorithm by taking $k = 6$ and $n = 3$. Consider the initial distribution, and right after the first step we have matrix A_0 (see below) We find :

P = 6296 , S_1= 6285 , S_2 = 6070 , S_3= 6535 ; $S_1, S_2, S_3 \in [6070 , 6535]$; j_m = 2, j_M =3 , d_{j_m} = –226 , $d_{j_M} = 239$, $\Delta* = 232$.

Next we have to find matrix B as follows:

$$A_0 = \begin{bmatrix} 1800 & 1850 & 2000 \\ 1800 & 1700 & 1600 \\ 1200 & 1200 & 1500 \\ 990 & 925 & 925 \\ 495 & 395 & 300 \\ 0 & 0 & 210 \end{bmatrix}$$

$$B = \begin{bmatrix} 150 & 300 & 800 & 1075 & 1605 & 2000 \\ -250 & -100 & 400 & 675 & 1205 & 1600 \\ -350 & -200 & 300 & 575 & 1105 & 1500 \\ -925 & -775 & -275 & 0 & 530 & 925 \\ -1550 & -1400 & -900 & -625 & -95 & 300 \\ -1640 & -1490 & -990 & -715 & -185 & 210 \end{bmatrix}$$

Here we have $b* = b_{66} = 210$, $p* = 6$, $q^* = 6$, which means that the best possible exchange between two elements of rows 2 and 3 is $a_{62} = 0$ by $a_{63} = 210$. Thus are obtained the new values S_3 = 6325, and S_2 = 6290, and hence the new interval in which S_1, S_2, S_3 will be found to be [6285, 6325].

Let us now look for a possibility to improve on exchanges of pairs of elements. Starting with b_{11} = 150 we begin by adding with $b_{22} = -100$, (as we must observe $p_1 < p_2$, $q_1 \neq q_2$). The result is 50, which is less than Δ* = 232. Next we move to the right and add up b_{11} with b_{23} : $b_{11}+b_{23}$ = 150 + 400 = 550 . Since the new value is greater than Δ* , we move downwards: 150 + 300 = 450 . It still is greater than Δ*. Next we calculate 150 – 275 = – 125 < Δ*. Now we must move to the right: 150 + 0 = 150 . This sum is less than Δ* , so that the next one is 150 + 530 = 680 > Δ*; next 150 – 95 = 55 < Δ* ; 150 + 300 = 450 > Δ* ; 150 + 210 = 360 > Δ*. The best we have accomplished so far is $b_{11}+b_{66}$ = 360 , but since this is a worse result than $b^* = b_{66}$ =210, there is no need to recall it.

Next we repeat the same procedure with b_{12} = 300: 300 – 250 = 50 < Δ* ; 300 + 400 = 700 > Δ* ; 300 + 300 = 600 > Δ* ; 300 – 275 = 25 < Δ* ; 300 + 0 = 300 > Δ* ; 300 – 625 = –325 < Δ* ; 300 – 95 = 205 < Δ* ; 300 + 300 = 600 > Δ* ; 300 + 210 = 510 . This is done with all the remaining elements of the first row, always trying to improve on the value b^* .

Now let us move to the second row Starting with $b_{21}+b_{32}$ (recall that $p_1 < p_2$, $q_1 \neq q_2$). After going over all possible sums with b_{21}, we move to $b_{22}, b_{23}, b_{24}, b_{25}$. Let us make a stop here: $b_{25}+b_{31}$ > Δ* ; $b_{25}+b_{41}$ < Δ* ; $b_{25}+b_{42}$ > Δ* ; $b_{25}+b_{52}$ < Δ* ; $b_{25}+b_{53}$ > Δ* ; $b_{25}+b_{63}$ = 215 . his value is better than b^* = 210, as it is closer to Δ*, therefore we have a better exchange than the indicated by $b^* = b_{66}$. In order to terminate the procedure with the element b_{25} we have to move to the right-hand side, since 215 < Δ* , then we find $b_{25} + b_{64}$ > Δ*, which would compel us to move downwards, but, we find ourselves on the last row of B , so that we en up with b_{25}. Continuing the procedure on table B, we see that b^{**} = $b_{25}+b_{63}$ = 215. This means that we must exchange the second and sixth elements of column j_M =3 by the fifth and third elements, respectively, on column j_m = 2.

Once this has been accomplished, we return to step one, and after repeating the procedure, the matrix will look as follows:

$$A_1 = \begin{bmatrix} 1800 & 1850 & 2000 \\ 1800 & 1700 & 1500 \\ 1200 & 1600 & 1200 \\ 990 & 925 & 925 \\ 495 & 210 & 395 \\ 0 & 0 & 300 \end{bmatrix}$$

Now S_1 = 6285 , S_2 = 6285 , S_3 = 6320 , j_m = 1, j_M =3 , $d_{j_m} = -11$, d_{j_M} = 24 , Δ* = 17.

Notice that there are two equal sums, so that, if we wish to render our task more orderly, as it were, we take j_m to be the least of the numbers on the columns where the value of the sum is the least. Hence we compare the first and the third columns. Table *B* will then look as follows:

$$B = \begin{bmatrix} 200 & 200 & 800 & 1010 & 1505 & 2000 \\ -300 & -300 & 300 & 610 & 1005 & 1500 \\ -600 & -600 & 0 & 210 & 705 & 1200 \\ -875 & -875 & -275 & 65 & 430 & 925 \\ -1405 & -1405 & -805 & -595 & -100 & 395 \\ -1500 & -1500 & -900 & -690 & -195 & 300 \end{bmatrix}$$

Notice that there is no b_{pq} in the interval [0,35], hence there is no permissible exchange between the elements of the first and third columns. However, going along table *B* by reckoning sums of the type $b_{p_1q_1}+b_{p_2q_2}$ (with $p_1 < p_2$, $q_1 \neq q_2$), we do find several permissible exchanges, for instance: $b_{11} + b_{65} = 5$. Out of all of them, the best one is $b_{34} + b_{65} = 210 - 195 = 15$.

After exchanging a_{33} y a_{41} by a_{63} y a_{51}, we get:

$$A_2 = \begin{bmatrix} 1800 & 1850 & 2000 \\ 1800 & 1700 & 1500 \\ 1200 & 1600 & 990 \\ 1200 & 925 & 925 \\ 300 & 210 & 495 \\ 0 & 0 & 395 \end{bmatrix}$$

Now $S_1 = 6300$, $S_2 = 6285$, $S_3 = 6305$, $j_m = 2$, $j_M = 3$, $d_{j_m} = -11$, $d_{j_M} = 9$, $\Delta^* = 10$.

In the process of writing table B we see that there is no b_{pq} within the interval of permissible exchanges, i.e., within $(0,20)$:

$$B = \begin{bmatrix} 150 & 300 & 400 & 1075 & 1790 & 2000 \\ -350 & -200 & -100 & 575 & 1290 & 1500 \\ -860 & -710 & -610 & 65 & 780 & 990 \\ -925 & -775 & -675 & 0 & 715 & 925 \\ -1355 & -1205 & -1105 & -430 & 285 & 495 \\ -1655 & -1305 & -1205 & -530 & 185 & 395 \end{bmatrix}$$

However, $b^{**} = b_{32} + b_{45} = -710 + 715 = 5$. Next we make the indicated exchanges: a_{33} and a_{43} by a_{22} and a_{52} , After the first step of the algorithm we obtain

$$A_3 = \begin{bmatrix} 1800 & 1850 & 2000 \\ 1800 & 1600 & 1700 \\ 1200 & 990 & 1500 \\ 1200 & 925 & 495 \\ 300 & 925 & 395 \\ 0 & 0 & 210 \end{bmatrix}$$

Here $S_1 = 6300$, $S_2 = 6290$, $S_3 = 6300$, $j_m = 2$, $j_M = 1$, $d_{j_m} = -6$, $d_{j_M} = 4$, $\Delta^* = 5$ is sufficient to get table B for $j_m = 2$, $j_M = 1$, and accordingly there are no changes to be made.

Next the two columns $j_m = 2$, $j_M = 3$ are compared. We repeat the same procedure and, after finding B , we likewise conclude that there are no possibilities of exchanges either. Therefore the procedure has come to an end and the best distribution happens to be A_3 .

5 General Remarks

1. The initial distribution may be thoroughly arbitrary, because the algorithm itself always looks, as it were, for the most even distribution. Indeed, it may happen that the "worst" distribution could pass into the "best" one in a single step, i.e., with a single exchange.

2. to be sure, the general method described here could give rise to many unnecessary calculations should n happen to be too large., as in that case we would have to analyze $n(n-1) / 2$ tables of the B type Whence it is convenient to set forth the alternative of stopping the procedure if $\Delta \le M$, for certain maximal value of M which would account for the greatest permissible difference between S_{j_m} and S_{j_M} , depending, of course, on the user's interests and preferences, or else to halt the procedure if the relative error $(S_{j_M} - S_{j_m})/ P$, becomes less than a certain $\varepsilon > 0$ previously given. In the above example, at the end it is readily seen that the relative error is $(S_{j_M} - S_{j_m}) / P \le 0{,}0017$.

3. At a cursory glance it may seem that the analysis of the B tables could be too cumbersome under certain circumstances. In fact, this is not the case, as owing to condition (3) the number of sums to carry out may not be greater than

$$k[(k-1)+k+(k+1)+...+(2k-3)] = k(k-1)(3k-4)/ 2$$

in each table. If in addition we take into account the fact that the operations to be performed in the algorithm are too elementary, it becomes clear that, all in all, it is a rather simple algorithm as compared with other known algorithms of linear programming. .

6 Conclusions

1. Our own experience has shown great effectiveness in the use of this algorithm. In the above example it was readily seen that right after the first step the length of interval of sums variation S_j got reduced from 465 to 35, reaching finally the value of 10 and only after three steps. In particular, we found the algorithm all too useful when dealing with boards of electric devices, as stated in the introduction.

2. While the original problem that gave rise to the whole idea was performed on a 10×3 arrangement, the algorithm is supposed to be useful for any number of rows and columns. Therefore, we consider this effort a useful alternative for certain minimax problems in integers.

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