On a rational mapping of a polynomial system into a quadratic system

KONSTANTIN E. STARKOV
CITEDI-IPN
Avenida del Parque 1310, Mesa de Otay, Tijuana 22510, B.C.
MEXICO

Abstract: - In this paper we prove that a polynomial system can be always mapped by some rational mapping into a general quadratic system. This result is useful for studies of systems with a complex dynamics. One generalization is discussed. Our approach is based on theorems on algebraic dependent polynomials.

Key-Words:- Polynomial, polynomial system, quadratic system, rational mapping, algebraic dependence, complex dynamics

1 Introduction

In this paper we consider a polynomial system

$\dot{x} = f(x),$ (1)

$x = (x_1, ..., x_n)^T \in \mathbf{R}^n$, with $n \geq 2$. We address to the following problem. Can a system (1) be mapped into a general quadratic system

$\dot{X} = q(X) + CX + D$ (2)

defined in a linear vector space $\mathbf{R}^N$ for some dimension N; here q is a homogeneous quadratic vector field ($q(\lambda X) = \lambda^2 q(X)$ for all $\lambda \in \mathbf{R}$ and all $X \in \mathbf{R}^N$); C is a $(N \cdot N)$ real matrix; $D \in \mathbf{R}^N$?

The reason of the interest to this question is related to the idea to consider a general quadratic system as a system lifted in a suitable algebra. This approach is fruitful for studies of different topics of qualitative theory of ordinary differential equations: periodic orbits, domains of attractions etc. See detailed discussions on this subject in the paper of Kinyon and Sagle, [5], and in the book of Walcher [9].

It is shown in [5], Proposition 2.1, that if

$z^{(n)} = p(z, \dot{z}, ..., z^{(n-1)})$ (3)

is a n- order differential equation, with p be a polynomial of its arguments, then the solution to this equation may be obtained from some solution of a quadratic system $\dot{Z} = Z^2$ presenting in a suitable algebra. Here the main idea of the proof consists in constructing of a monomial mapping which maps (3) into a general quadratic system (2) living in some linear vector space $\mathbf{R}^N$. We prove that the answer for our question is always affirmative if we allow that this mapping is rational and is well defined outside some algebraic set strictly contained in the state space R^n. The structure of this paper is as follows. Section 2 contains the formulation of two classical theorems on algebraically dependent polynomials which are used in the proof. The main result of this paper (Theorem 3) is presented in Section 3. One special case is also studied there. We give one remark concerning an application of Theorem 3 to the case of Noetherian systems in Section 4.

2 Some algebraic preliminaries

Let $\mathbf{R}[X] = \mathbf{R}[x_1,...,x_n]$ and $\mathbf{R}[W] = \mathbf{R}[w_1,...,w_{n+1}]$ be two rings of polynomials. We remind two classical results about algebraically dependent polynomials used in this paper.

Theorem 1. (Perron, [6]; see the modern version in [7]). Let $F_1(x_1,...,x_n),...,F_n(x_1,...,x_n),F_{n+1}(x_1,...,x_n)\in$ **R**[X] be polynomials of positive degrees of variables $(x_1,...,x_n)$ Let v be the weight of the ring **R**[W] defined by conditions $v(w_i)$ = deg F_i for i =1, ..., n + 1. Then there is a nontrivial polynomial T $\in$ **R**[W], W = $(w_1, ...,w_{n+1})$ such that

$$T(F_1(x_1, ..., x_n), ..., F_{n+1}(x_1, ..., x_n)) \equiv 0,$$

with $v(T) \leq \prod_{i=1}^{n+1} \deg F_i$.

Theorem 2. (Gordan, [2], see the modern version in [3]). If two polynomials $P(x_1, ..., x_n)$ and $Q(x_1, ..., x_n)$ (being nonconstant polynomials) are algebraically dependent then one can find a polynomial $W(x_1, ..., x_n)$ and two polynomials of one variable p(t), q(t) such that $P(x_1, ..., x_n) = p(W(x_1, ...,x_n))$ and $Q(x_1, ..., x_n) =q(W(x_1, ..., x_n))$.

3 Main results

By $\varphi(x, t)$ we denote a solution of the system (1) such that $\varphi(x, 0) = x$. Let L_f be a Lie derivative along the vector field f and $L_f^i\psi=L_f(L_f^{i-1}\psi)$, $i \geq 1$, $L_f^0\psi=\psi$ for any $\psi \in$ **R**[X].

The main result of this paper is contained in

Theorem 3. One can find an algebraic set A defined as $A=a^{-1}(0)$ for some real nonzero polynomial $a(x_1,\ldots,x_n)$ or $A=\phi$, and a system (2) defined on a linear vector space R^N for some integer N such that there is a rational mapping $\Psi : \mathbf{R}^n \to \mathbf{R}^N$ with following properties: i) Ψ is well-defined on $\mathbf{R}^n$ -A, ii) if $x \notin A$ then $\Psi(\varphi(x, t)) = \Phi(\Psi(x, t))$. Here by $\Phi(X, t)$ we denote a solution of (2) with an initial condition $\Phi(X, 0) = X$.

Proof. We take any polynomial $\psi\in$ **R**[X] such that $L_f^n\psi$ is nonconstant. Let $z = \psi(x)$. According with Theorem 1 there is a polynomial $G \in \mathbf{R}[w_1, ...,w_{n+1}]$ such that $G(\psi(x), L_f\psi(x), ..., L_f^n \psi(x)) \equiv 0$.

Without loss of genericity we consider that the polynomial G with this property has a minimal possible degree with respect to variables $(w_1, ...,w_{n+1})$.

Now by borrowing one idea from the lemma in [8] we can express G as a sum of polynomials $G_1 + G_2$, where G_1 contains all monomials of G depending on w_{n+1} and does not contain any monomial of G which is not depended on w_{n+1};$G_2 = G-G_1$. Now if

$$G_1(x) = \sum_{\gamma} a_\gamma(\psi(x))^{\gamma_0}\ldots(L_f^n\psi(x))^{\gamma_n}$$

then

$$L_fG_1(x) = L_f^{n+1}\psi(x) \cdot Q(\psi(x), L_f\psi(x), ...,L_f^n\psi(x)) +B(\psi(x), L_f\psi(x), ..., L_f^n \psi(x))$$

for some polynomials B,Q, with

$$v(Q) < v(G_1) \leq v(G). \tag{4}$$

It follows from similar computations for L_fG_2 that

$$L_f^{n+1}\psi(x) \cdot Q(\psi(x), L_f\psi(x), ..., L_f^n\psi(x)) +P(\psi(x), L_f\psi(x), ...,L_f^n\psi(x)) \equiv 0 \tag{5}$$

for some polynomial P. Therefore the expression $Q(\psi(x), L_f\psi(x), ...,L_f^n\psi(x))$ is not identically equal to zero because of (4). We denote its set of zeroes in $\mathbf{R}^n$ by A. It follows from (5) that by use of the mapping $\Psi: \mathbf{R}^n \to \mathbf{R}^{n+1}$, $\Psi(x) =(\psi(x), L_f\psi(x), ..., L_f^n\psi(x))^T$, we map the system (1) into the system $z^{(n+1)}=-P(z, \dot{z}, ..., z^{(n)})/Q(z, \dot{z}, ..., z^{(n)})$ defined outside of the set $Q^{-1}(0)$. Now by use $\omega = Q^{-1}(z, \dot{z}, ..., z^{(n)})$ we compute $\dot{\omega} = \dot{Q}/Q^2 = -\omega^2Q_1-\omega^3PQ_2$ for some polynomials Q_1;Q_2 depending on $z, \dot{z}, ..., z^{(n)}$.

As a result, we get a new system

$$z^{(n+1)} = -\omega P(z, \dot{z}, ..., z^{(n)}), \qquad (6)$$

$$\dot{\omega} = -\omega^2 Q_1(z, \dot{z}, ..., z^{(n)})$$

$$-\omega^3 P(z, \dot{z}, ..., z^{(n)}) Q_2(z, \dot{z}, ..., z^{(n)})$$

and let F be a vector field of (6) written as a system of the first order in (ω,Z)-coordinates, with $Z = (z_1, ..., z_{n+1})^T$ and $z_1 = z$. Let L_F be corresponding Lie derivative.

Further, for any polynomial

$$h(x_1, ..., x_n) = \sum_a h_a x^a, \; x^a = x_1^{a_1} \ldots x_n^{a_n}, \qquad (7)$$

we introduce the set

supp h = $\{\alpha \mid h_\alpha \neq 0$ in the formula (7)$\}$.

We define following sets of multiindices: Ω_0 = suppP; Ω_1 = suppQ_1; Ω_2 = suppPQ_2. By e_s, s = 1, ..., n + 1, we denote the standard orthonormal basis in $\mathbf{R}^{n+1}$. Let $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2 \cup (\cup_{s=1}^{n+1} e_s)$. By $U(\Omega_2)$ we denote the set of monomials with degrees from the set Ω_2.

Then we define a monomial mapping by formulae

$$u_{s\alpha} = \omega^s Z^\alpha; \; \alpha \in \Omega, \; s = 0, 1, 2; \qquad (8)$$

$u_{10} = \omega$; here the index $0 \in R^{n+1}$.

By $U(\Omega)$ we denote the set of monomials given in (8). Also, we introduce $|\alpha| = \sum_{j=1}^{n+1} \alpha_j$.

Now our goal is to enlarge the set Ω up to some set Ω_* for which the corresponding set $U(\Omega_*)$ of monomials forms the mapping $\Psi := \Psi(\Omega_*)$ such that Ψ maps (1) into some system (2).

Firstly, we take

$$L_F\omega = -\omega(\omega Q_1) - \omega(\omega^2 PQ_2).$$

Since both Q_1 and PQ_2 belong to the linear span of monomials from $U(\Omega_2)$ the last formula gives a quadratic differential equation expressed with help of monomials of $U(\Omega)$.

Further, we examine two cases: $L_F Z^\alpha$, $\alpha \in \Omega$, contains 1) a monomial Z^β, and 2) a monomial $Z^\beta \omega$. In the first case $|\beta| = |\alpha|$ up to a nonzero coefficient . In the second case $\beta = \alpha - e_{n+1} + \gamma$, with $\alpha_{n+1} \geq 1$, for some $\gamma \in \Omega_0$. Now if Z^β ($Z^\beta \omega$ correspondingly) is not contained in $U(\Omega)$ then we add it into $U(\Omega)$. After this in accordance with these two cases we take $L_F Z^\beta$ and $L_F Z^{\beta-\gamma}$, with $|\beta-\gamma| = |\alpha| - 1$ and we repeat our arguments. Since the number of monomials Z^δ, with $|\delta| \leq |\alpha|$ is finite, we add on this way only a finite number of monomials into $U(\Omega)$. Now we repeat this argument for each multiindex $\alpha \in \Omega$. As a result, we obtain some completed set of monomials defined below by $U(\Omega_*)$ for some set Ω_* of multiindices, with $\Omega_* \supset \Omega$.

At last, we take any $\omega^s Z^\alpha$ chosen from the set $U(\Omega_*)$, s = 1, 2. We can write

$$L_F(\omega^s Z^\alpha) = -s(\omega Q_1) \cdot (\omega^s Z^\alpha) - \qquad (9)$$

$$s(\omega^s Z^\alpha) \cdot (\omega^2 PQ_2) + (\omega^s L_F Z^\alpha);$$

$$s = 1, 2.$$

Here in notations introduced above we have the following. Let s = 1. Then the expression in the last parenthesis in (9) can contain only monomials of types ωZ^β and $(\omega Z^{\alpha - e_{n+1}}) \cdot (\omega Z^\gamma)$, with ωZ^β ; $\omega Z^{\alpha - e_{n+1}}$; ωZ^γ contained in $U(\Omega_*)$. Now let s = 2. Then the expression in the last parenthesis in (9) can contain only monomials of types $\omega^2 Z^\beta$ and $(\omega Z^{\alpha - e_{n+1}}) \cdot (\omega^2 Z^\gamma)$, with $\omega^2 Z^\beta$; $\omega Z^{\alpha - e_{n+1}}$; $\omega^2 Z^\gamma$ contained in $U(\Omega_*)$. By examining other expressions in parenthesis in the right- side of (9), we can see that each of them is contained in the linear span of monomials from $U(\Omega_*)$. Therefore (9) gives quadratic differential equations expressed with help of monomials of $U(\Omega_*)$.

Finally, we note that, by construction, N = card($U(\Omega_*)$). Also, if a is the maximal degree of monomials entering into the vector-polynomial F then

$$N \leq 1 + 3\sum_{s=0}^{a} \binom{n+s}{s}.$$

Hence, the desirable assertion is proved.

The bound for N contained in this theorem looks high in comparison with the dimension

n. But sometimes the real value for N can be less than n for a clever choice of a polynomial Ψ.

Example 1. Consider a system

$$\begin{aligned}\dot{x}_1 &= -x_2 + x_1(1 - x_1^2 - x_2^2 - x_3^2);\\ \dot{x}_2 &= x_1 + x_2(1 - x_1^2 - x_2^2 - x_3^2);\\ \dot{x}_3 &= x_3(1 - x_1^2 - x_2^2 - x_3^2)\end{aligned} \qquad (10)$$

and a function $\Psi = x_1^2 + x_2^2 + x_3^2$. Then by computation of $L_f \Psi$ we establish that this system is mapped into the one-dimensional system $\dot{z} = 2z(1-z)$. Now we examine one special situation on this matter.

Proposition 4. Suppose that we can find a polynomial g with following properties: 1) if g(x) = p(W(x)) is some representation of g as a composition of two polynomial mappings W : $\mathbf{R}^n \to \mathbf{R}^1$, p :$\mathbf{R}^1 \to \mathbf{R}^1$, then it follows that p is linear; 2) polynomials {g, ..., L_f^{i-1} g} are algebraically independent in **R**[X] while g and L_f^i g are algebraically dependent in **R**[X], i ≤ n. Then there exist a polynomial mapping Ψ: $\mathbf{R}^n \to \mathbf{R}^i$ and an i-dimensional polynomial system

$$\dot{z}_1 = z_2, ..., \dot{z}_{i-1} = z_i, \dot{z}_i = F(z_1) \qquad (11)$$

such that $\Psi(\varphi(x, t)) = \Phi(\Psi(x), t)$ on $\mathbf{R}^n$, with Φ be a solution of (11), $\Phi(z, 0) = z \in \mathbf{R}^i$.

Proof. It follows from Theorem 2 applied to polynomials g and L_f^i g and the first condition that one can find polynomials g : $\mathbf{R}^n \to \mathbf{R}^1$ and W : $\mathbf{R}^n \to \mathbf{R}^1$, such that L_f^i g = F(g) for some polynomial F. This fact implies the result of Proposition 4, with Ψ:x → (g(x), L_f g(x), ..., L_f^{i-1} g(x)).

Consider the case: n=2. We introduce a mapping Ξ : x→ (g(x), L_f g(x)). Then, as it directly follows from [4], Proposition 2, g and L_f g are algebraically dependent in **R**[X] if and only if the jacobian of Ξ is identically equal to zero.

Remark 1. Suppose that conditions of Proposition 4 are held. In this case localization problems for attraction domains and for limit sets of the system (1) are solved in terms of preimages of equilibria of the system (11) respecting to the mapping g.

For example, let us take the system (10). We have: $\Psi^{-1}(0) = 0$ which is an equilibrium of the system (10), $\Psi^{-1}(1)= \{x_1^2+x_2^2+x_3^2=1\}$ which is an invariant two- dimensional sphere being an attractor of the system (10).

Corollary 5. Assume that (1) is a general cubic system, n=2 and we can find a quadratic form g satisfying conditions of Proposition 4. Then, by application of Proposition 4, the system (11) is a general quadratic system.

4 Remark on Noetherian systems

Firstly, we note that all results remain valid for (1) defined in the complex state space $\mathbf{C}^n$ because Theorem 1;2 are true for complex polynomials. Also, we remind, see e.g. [1], that a ring K of analytic functions defined on an open domain $U \subset \mathbf{C}^n$ is called a ring of Noetherian functions in U if 1) K contains the polynomial ring **C**[X] of polynomials of n variables and is finitely generated over this ring; 2) K is closed under differentiation. A set of functions $\chi = \{\chi_1, ..., \chi_m\}$ is called a Noetherian chain of order m if these functions generate K over **C**[X]. We shall call a system of a type (1) Noetherian if each component f_s is expressed as $\rho_s(x, \chi(x))$ for some polynomial ρ_s, s = 1, ..., n. We define the mapping U → $\mathbf{C}^n \times \mathbf{C}^n$ by the formula x → (x, χ (x)). Then it is clear that the Noetherian system is mapped by this mapping into some polynomial system of dimension n + m. Therefore, by Theorem 3, it is also mapped into some general quadratic system. For example, consider a Li´enard equation $\dot{x}_1 = x_2 - \alpha \sin x_1$; $\dot{x}_2 = -x_1$. Since {sin x_1; cos x_1} forms a Noetherian chain then by use the mapping $z_1 = x_1$; $z_2 = x_2$; $z_3 = \sin x_1$; $z_4 = \cos x_1$ we map a Li´enard equation into $\dot{z}_1 = z_2 - \alpha z_3$; $\dot{z}_2 = - z_1$; $\dot{z}_3 = z_2 z_4 - \alpha z_3 z_4$; $\dot{z}_4 = -z_2 z_3 + \alpha z_3^2$.

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