Fracture dynamics with crack edges contact interaction

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Abstract: - In this paper we consider classical and variational of fracture dynamic problems with taking into account a contact interaction of the crack edges. An algorithm for the solution of the elastodynamic contact problem with unilateral constrain has been developed. It arises from the boundary variational principle of elastodynamics for bodies with cracks and unilateral constrains on their sides. Some numerical results are presented for 3-D elastodynamic unilateral contact problem for bodies with cracks.

Key-Words: - Fracture, elastodynamics, unilateral contact, variational principle, stress intensity factor.

1 Introduction

In many situations in structural and material design that use methods of fracture mechanics, the inertial effects resulting from dynamic load and crack propagation need to be taken into account. Analysis of fracture mechanics problems demonstrates that taking the crack edge contact interaction into account may significantly affect the fracture mechanics criteria.

In the case of harmonic loading, the problem of crack edge contact interaction is very important. Under the action of an harmonic load, the steady-state condition is usually considered but the possibility of crack edges contact interaction is not counted. In this case, the dependence of all functions, which determine the stress-strain state of a body with cracks, is harmonic in time. For example, using such approach, the problem of the interaction of a tension-compression harmonic wave with a crack of finite length in a plane has been solved in [3]. In this and other publications it has been mentioned that such approach to this problem is not correct, as it does not take into account the crack edge contact interaction that always occurs during the action of the compressive wave. A critical analysis of the problem has been reported in [4-6].

Under load action, existing cracks in the material form interfaces at the crack surfaces with the unilateral contact forces at the normal direction and the frictional contact forces at the tangential direction. The unilateral contact and friction forces are non-linear phenomena because, *a priori,* the boundaries between contact and non-contact regions, and also between adhesion and sliding zones, are unknown. Mathematically such boundary conditions are formulated in form of inequalities. Mathematical formulation of various problems with constrains in form of inequalities, which aroused in physics and engineering have been considered in [1, 2, 7]. Important for application in fracture mechanics elastodynamic contact problems with unilateral constrains and friction for bodies with cracks have been investigated in [4-6, 14].

The main difficulty in elastodynamic problem with unilateral constrains and friction is the boundary conditions in form of inequalities, which makes the problem nonlinear. This nonlinearity is specific. It leads to nonsmooth functionals if the problem is formulated using variational technique [7]. In order to mathematically investigate such problems, new theoretical and numerical methods have been developed [2, 7]. Some new formulations and algorithms for the problem solution based on variational principles of elasticity have been developed by Zozulya in [8-10] and some mathematical aspects of the elastodynamic contact problem with friction have been investigated in [11, 15].

Usually unilateral contact problems with friction are solved numerically using finite element [2, 7] or boundary element [1, 4-6, 14] methods. In this paper the boundary element method (BEM) will be used for the solution of the unilateral contact problem with friction in fracture mechanics. Some helpful details of a hypersingular integral regularization, which arise during the application of the BEM in fracture mechanics, may be found in [12, 13].

In this paper some new results of mathematical formulation and numerical methods of elastodynamic problems for bodies with cracks with allowance for their edge contact interaction are presented. The mechanical effects caused by contact interaction and their influence on the fracture mechanics parameters were investigated for penny-shaped cracks.

2 Classical formulation of the problem

Let us consider an elastic body in three-dimensional Euclidean space R^3 that occupies a volume V. The boundary of the body ∂V is a piecewise smooth and consists of two parts: ∂V_u, where displacements vector $\mathbf{u}(\mathbf{x},t)$ is assigned and ∂V_p, where surface traction vector $\mathbf{p}(\mathbf{x},t)$ is assigned respectively. There is an arbitrary oriented crack in the body, which is described by its surfaces $\Omega^+ \cup \Omega^-$, where Ω^+ and Ω^- are opposite edges. The body may be affected by body forces $\mathbf{b}(\mathbf{x},t)$.

Linear equations of the elastodynamics in form of displacement with mixed boundary conditions and the initial conditions may be present in the form

$$A_{ij}u_j + b_i = \rho\partial_t^2 u_i, \forall \mathbf{x} \in V, \forall t \in \Im = [t_o, t_1],$$
$$A_{ij} = \mu\delta_{ij}\partial_k\partial_k + (\lambda+\mu)\partial_i\partial_j \quad (1)$$
$$p_i(\mathbf{x},t) = \psi_i(\mathbf{x},t), \forall\, \mathbf{x} \in \partial V_p, u_i(\mathbf{x},t) = \varphi_i(\mathbf{x},t), \forall\, \mathbf{x} \in \partial V_u$$
$$u_i(\mathbf{x},t_o) = u_i^0(\mathbf{x}) \;,\; \partial_t u_i(\mathbf{x},t_o) = v_i^0(\mathbf{x}) \;,\; \forall\, \mathbf{x} \in V$$

In these formulas the following indications are introduced: $\partial_i = \partial/\partial x_i$ and $\partial_t = \partial/\partial t$ are derivatives with respect to coordinates and time respectively, λ and μ are the Lame constants. The summation convention applies to repeated indices.

At the contact surfaces the one sided constrains in the form of inequalities must be satisfied [2, 7]

$$\Delta u_n \geq -h_o, q_n \geq 0, (\Delta u_n + h_o)q_n = 0, \forall \mathbf{x} \in \Omega, \forall t \in \Im \quad (2)$$

where q_n and Δu_n are the normal components of contact forces and displacement discontinuity vectors respectively, h_0 an initial distance between contacting surface, Ω is a crack surface (see [4-6] for details).

We assume that crack surfaces are smooth and not friction take place during their contact interaction.

3 Boundary variational principle

Let us consider a variational formulation of the elastodynamic contact problem with unilateral constrains and friction for elastic bodies with cracks in the form of boundary variational inequalities. Following [8] a boundary variational functional $\Phi[u_i, p_i]$ may be obtained in the form

$$\Phi[u_i, p_i] = \tfrac{1}{2} < p_i, u_i >_{\partial V \times \Im} +$$
$$+ < \tfrac{1}{2} p_i - \psi_i, u_i >_{\partial V_p \times \Im} + < \tfrac{1}{2} u_i - \varphi_i, p_i >_{\partial V_u \times \Im} \quad (3)$$

where $<\cdot,\cdot>$ denotes the duality pairing for Banach space $\mathbf{H}^{1/2,1/2}(\partial V \times \Im)$ and its dual functional space $\mathbf{H}^{-1/2,-1/2}(\partial V \times \Im)$. For details see [8-11].

The functional (3) has to be considered on the set of admissible displacements of the form:

$$\mathbf{K}[u_i, p_i] = \{ u_i \in \mathbf{H}^{1/2,1/2}(\partial V \times \Im),$$
$$p_i \in \mathbf{H}^{-1/2,-1/2}(\partial V \times \Im), \varepsilon_{ij}(\mathbf{u}) = \tfrac{1}{2}(\partial_i u_j + \partial_j u_i),$$
$$\sigma_{ij}(\mathbf{u}) = c_{ijkl}\varepsilon_{kl}(\mathbf{u}), \partial_j \sigma_{ij} + b_i = \rho\partial_t^2 u_i,$$
$$u_i(\mathbf{x},t_o) = u_i^0(\mathbf{x}), \partial_t u_i(\mathbf{x},t_o) = v_i^0(\mathbf{x}) \;, \forall\, t \in \Im\} \quad (4)$$

Then the initial-boundary value problem (1) with the unilateral constrains (2) will be transformed into variational form. In [10] it has been shown that the problem may be formulated in one of the following forms

Find $u_i, p_i \in \mathbf{K}_c(u_i, p_i)$ *such that*
$\Phi[u_i, p_i] = \inf\limits_{u_i^*, p_i^*, q_i^* \in \mathbf{K}_c(u_i,p_i)} \sup \{\Phi[u_i^*, p_i^*] - < q_i^*, u_i >_{\Omega\times\Im}\}$
where $\mathbf{K}_c(u_i, p_i) = \mathbf{K}(u_i, p_i) \cap \mathbf{K}_n(u_n)$

or (5)

Find $u_i, p_i \in \mathbf{K}_{c,n}^*(u_i, p_i)$ *such that*
$\Phi[u_i, p_i] = \inf\limits_{u_i^*, p_i^*, u_i^* \in \mathbf{K}_c^*(u_i,p_i)} \sup \{\Phi[u_i^*, p_i^*] - < q_i, u_i^* >_{\Omega\times\Im}\}$
where $\mathbf{K}_c^*(u_i, p_i) = \mathbf{K}(u_i, p_i) \cap \mathbf{K}_n^*(q_n)$

where:

$$\mathbf{K}_n(u_n) = \{u_n \in \mathbf{H}^{1/2,0}(\Omega \times \Im), u_n - h_0 \geq 0\}, \forall \mathbf{x} \in \Omega\}$$
$$\mathbf{K}_n^*(\sigma_n) = \{\sigma_n \in \mathbf{H}^{-1/2,0}(\Omega \times \Im), \sigma_n \geq 0, \forall \mathbf{x} \in \Omega\} \quad (6)$$

These variational formulations have been used by Zozulya in [8,10] to develop different algorithms for solution of elastodynamic problems with unilateral constrains and friction.

4 Algorithms for the problem solution

There are many algorithms, which may be used for solution of the elastodynamic contact problems with one sided constrains (1) and (2). For references see [1, 4-6,7]. Most algorithms consist of two parts. The first one finds the solution of the variational problem (5) on the set (4) without constrains. Then in second part of the algorithm the functional (5) is corrected using the values of q_n and Δu_n which were in the first part of the algorithm. Usually, it is necessary to perform several steps of iterations in order to solve the problem. Only the second part of the algorithm, the operator of orthogonal projection on the sets of constrains, is well determined. For example, in the elastodynamic unilateral contact problem for bodies with cracks under consideration it is a projection on the set of unilateral constrains (6). Here it will be used the algorithm which is slightly different from the one developed in [8] and used in [4-6,12].

The algorithm consists of the following steps:

a) the initial distribution of the contact forces on the contact surface $q_n^0(\mathbf{x},t)$, $\forall \mathbf{x} \in \Omega$, $\forall t \in \Im$ is assigned;

b) the problem without constrains is solved and the unknown quantities on the region and/or on the boundary and also on the contact surfaces $\Delta u_n(\mathbf{x},t)$ are defined;

c) the normal components of the displacements vector $\Delta u_n(\mathbf{x},t)$ are corrected to satisfy the unilateral restrictions:

$$\Delta u_n^{k+1}(\mathbf{x},t) = U_n[\Delta u_n^k(\mathbf{x},t)] \ ,$$

where $U_n[\Delta u_n] = \begin{cases} \Delta u_n & , \text{ if } q_n \le 0 \\ -h_o & , \text{if } q_n > 0 \end{cases}$,

d) the normal components of the vector of contact $q_n(\mathbf{x},t)$ forces are calculated;

e) the normal components of the vector of contact forces $q_n(\mathbf{x},t)$ are corrected to satisfy the constrains

$$q_n^{k+1}(\mathbf{x},t) = P_n[q_n^k(\mathbf{x},t)] \ ,$$

where $P_n[q_n] = \begin{cases} 0 & , \text{ if } q_n \le 0 \\ q_n & , \text{if } q_n > 0 \end{cases}$,

f) then proceed to the second step of the iteration.

Some other algorithms of this type have been developed in [10,11]. We will use here the boundary integral equation (BIE) method as the first part of the algorithm.

5 Plane crack in 3-D elastic space

Lets a 3-D elastic unbounded body has a finite crack which is located in the plane $R^2 = \{\mathbf{x} : x_3 = 0\}$. An harmonic tension-compression wave propagates in the direction perpendicular to the surface of the crack Ω, as shown in Fig 1.

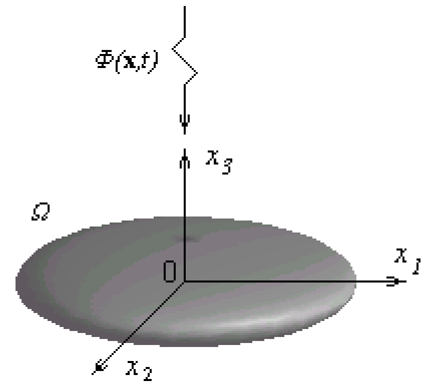

Fig. 1. Plane crack under the action of a harmonic wave of tension-compression.

The incident wave is defined by the potential function

$$\Phi(\mathbf{x},t) = \Phi_0 e^{i(k_1 x_3 - \omega t)} \ , k_1 = \omega / c_1 \ , c_1 = \sqrt{(\lambda + 2\mu)/\rho}$$

where ρ is a density of the material, $\omega = 2\pi/T$ is the frequency, T is the period of vibration, Φ_0 is the amplitude and c_1 is the velocity of the dilatational wave.

The determination of the stress-strain state under action of the incident wave is not a complicated problem, (see: [4-6]). Therefore lets consider the problem concerning reflected waves. The load on the crack edges caused by the incident wave has the form

$$p_3^*(\mathbf{x},t) = \mathrm{Re}\left\{ -k_1^2 \Phi_0 e^{i(k_1 x_3 - \omega t)} \right\}$$

The forces of contact interaction and displacement discontinuity on the crack surfaces should satisfy the unilateral constrains in the form of the inequalities (2).

The contact interaction of the crack edges makes the problem "constructively" nonlinear. This follows from the fact that, due to the contact interaction, the load on the crack edges has the form

$$p_3(\mathbf{x},t) = \begin{cases} p_3^*(\mathbf{x},t) + q_3(\mathbf{x},t), \forall \mathbf{x} \in \Omega_e, \\ p_3^*(\mathbf{x},t), \forall \mathbf{x} \notin \Omega_e, \end{cases}$$

Here $\Omega_e \subset \Omega$ is the region, where the crack edge contact interaction takes place. The size and form of this region is unknown beforehand. It changes in time and must be defined during solution of the problem.

As it was mentioned before, this problem is nonlinear. As a result, components of the stress-strain state caused by the reflected waves can not be represented as functions of coordinates **x**, multiplied by $e^{-i\omega t}$. That is why we will expand the components of the displacement discontinuity and traction vectors into a Fourier series with the loading parameter ω

$$p_3(\mathbf{x},t) = \mathrm{Re}\left\{ \sum_{k=-\infty}^{+\infty} p_3^k(\mathbf{x}) e^{i\omega_k t} \right\},$$

$$\Delta u_3(\mathbf{x},t) = \mathrm{Re}\left\{ \sum_{k=-\infty}^{+\infty} \Delta u_3^k(\mathbf{x}) e^{i\omega_k t} \right\},$$

where $\omega_k = 2\pi k/T$ and

$$p_3^k(\mathbf{x}) = \frac{\omega}{2\pi} \int_0^T p_3(\mathbf{x},t) e^{-i\omega_k t} dt,$$

$$\Delta u_3^k(\mathbf{x}) = \frac{\omega}{2\pi} \int_0^T \Delta u_3(\mathbf{x},t) e^{-i\omega_k t} dt.$$

The Fourier coefficients $p_3^k(\mathbf{x})$ and $\Delta u_3^k(\mathbf{x})$ satisfy the following BIEs

$$p_3^k(\mathbf{x}) = -f.p.\int_\Omega F_{33}(\mathbf{x},\mathbf{y},\omega_k) \Delta u_3^k(\mathbf{y}) d\Omega, \quad (7)$$

Here the kernels $F_{33}(\mathbf{x},\mathbf{y},\omega_k)$ are the fundamental solutions for the elastodynamic problems in a frequency domain. Following [12] we present them in the form

$$F_{33}(\mathbf{x},\mathbf{y},\omega_k) = \frac{1}{r^3}\frac{\mu(\lambda+\mu)}{2\pi(\lambda+2\mu)} - \frac{1}{r^3 4\pi\mu} \times$$
$$\times \left\{ 4\mu^2 \sum_{n=1}^{\infty} \frac{(-l_2)^n}{n!} \frac{(n-1)^2}{(n+2)} + \right.$$
$$\left. + \sum_{n=1}^{\infty} \left[\lambda^2 n(n+2) + 4\lambda\mu(n+2) + 12\mu^2 \right] \frac{(-l_1)^n}{n!} \frac{c_2^2}{c_1^2} \frac{(n-1)}{(n+2)} \right\}$$

Here $c_2 = \sqrt{\mu/\rho}$ is a distortional waves velocity and $l_1 = i\omega_k r/c_1$, $l_2 = i\omega_k r/c_2$, $r^2 = (y_i - x_i)(y_i - x_i)$.

These kernels are complex functions and may be presented in the form:

$$F_{33}(\mathbf{x},\mathbf{y},\omega_k) = F_{33}^{\mathrm{Re}}(\mathbf{x},\mathbf{y},\omega_k) + iF_{33}^{\mathrm{Im}}(\mathbf{x},\mathbf{y},\omega_k)$$

Therefore the BIEs (7) are complex-value. They establish relations between the complex functions $p_3^k(\mathbf{x})$ and $\Delta u_3^k(\mathbf{x})$. Also the BIEs (7) are hypersingular ones. Therefore the integrals in (7) should be considered in the sense of the finite part according to Hadamard, as it have been done in [12, 13].

Following [12] we rewrite the BIEs (7) in the form

$$p_{3,c}^k(\mathbf{x}) = -f.p.\int_\Omega F_{33}^{\mathrm{Re}}(\mathbf{x},\mathbf{y},\omega_k) \Delta u_{3,c}^k(\mathbf{y}) d\Omega - \int_\Omega F_{33}^{\mathrm{Im}}(\mathbf{x},\mathbf{y},\omega_k) \Delta u_{3,s}^k(\mathbf{y}) d\Omega$$

$$p_{3,s}^k(\mathbf{x}) = -f.p.\int_\Omega F_{33}^{\mathrm{Re}}(\mathbf{x},\mathbf{y},\omega_k) \Delta u_{3,s}^k(\mathbf{y}) d\Omega + \int_\Omega F_{33}^{\mathrm{Im}}(\mathbf{x},\mathbf{y},\omega_k) \Delta u_{3,c}^k(\mathbf{y}) d\Omega, \quad (8)$$

where

$$p_{3,c}^k(\mathbf{x}) = \frac{\omega}{\pi} \int_0^T p_3(\mathbf{x},t) \cos(\omega_k t) dt,$$

$$p_{3,s}^k(\mathbf{x}) = \frac{\omega}{\pi} \int_0^T p_3(\mathbf{x},t) \sin(\omega_k t) dt,$$

$$\Delta u_{3,c}^k(\mathbf{x}) = \frac{\omega}{\pi} \int_0^T \Delta u_3(\mathbf{x},t) \cos(\omega_k t) dt,$$

$$\Delta \mathrm{u}_{3,s}^k(\mathbf{x}) = \frac{\omega}{\pi} \int_0^T \Delta u_3(\mathbf{x},t) \sin(\omega_k t) dt.$$

Physical values of the $p_3(\mathbf{x},t)$ and $\Delta u_3(\mathbf{x},t)$ are expressed by the Fourier trigonometric series $\cos(\omega_k t)$ and $\sin(\omega_k t)$ in the form

$$p_3(\mathbf{x},t) = \frac{p_{3,c}^0(\mathbf{x})}{2} + \sum_{k=1}^{\infty} \left(p_{3,c}^k(\mathbf{x}) \cos(\omega_k t) + + p_{3,s}^k(\mathbf{x}) \sin(\omega_k t) \right)$$

$$\Delta u_3(\mathbf{x},t) = \frac{\Delta u_{3,c}^0(\mathbf{x})}{2} + \sum_{k=1}^{\infty} \left(\Delta u_{3,c}^k(\mathbf{x}) \cos(\omega_k t) + + \Delta u_{3,s}^k(\mathbf{x}) \sin(\omega_k t) \right)$$

The problem under consideration has been transformed into the BIEs (8), which has to be solved taking the unilateral constrains (2) into account.

The crack edges contact interaction influences the crack opening and the stress-strain state in the vicinity of the crack apex. Therefore it influences the parameters of fracture mechanics. The stress intensity factor (SIF) is the main parameter in linear fracture mechanics. As it was noted in [4, 6], more accurate results for the SIF calculation may be obtained with the following expressions

$$u_3(t) = \frac{K_I(t)}{\mu}\sqrt{\frac{r}{2\pi}}\sin\frac{\theta}{2}\left[1 - 2\nu + \sin^2\frac{\theta}{2}\right],$$

where r and θ are polar coordinates, connected with the apex of the crack, ν is a Poisson's ratio.

Expressions for the SIF will be obtained as limit transition from previous expression in the following form

$$K_I(t) = \lim_{r \to 0}\frac{\mu}{4(1-\nu)}\sqrt{\frac{2\pi}{r}}\Delta u_3(\mathbf{x},t)$$

This expression will be used in the numerical calculation of the SIF.

6 Numerical examples

The computational technique developed in the previous sections has been applied to solve of the problem of an harmonic loading of crack edges with considering their contact interaction.

Lets apply an harmonic load of unit amplitude to the crack edges. Assume that the material has the following mechanical properties: elastic modulus *E*=200 *GPa*, Poisson ratio ν=0.3 and specific density ρ=7800 kg/m^3. Let us consider a penny-shaped crack, which surface has coordinates

$$\Omega = \left\{x_1^2 + x_2^2 \le l, x_3 = 0\right\}$$

Here l is a crack radius.

Information of the contact forces distribution at the time $t = 4\pi/5\omega$ is presented at Fig. 2 and the displacement discontinuity at time $t = 0$ is presented at Fig. 3. From our calculation it follows that, at times when the displacement discontinuity $\Delta u_3(\mathbf{x},t) > 0$, the contact force $q_3(\mathbf{x},t) = 0$ or vice versa, when the contact force $q_3(\mathbf{x},t) > 0$, the displacement discontinuity $\Delta u_3(\mathbf{x},t) = 0$. The third of the unilateral conditions (2) takes place.

Many graphs of $\Delta u_n(\mathbf{x},t)$ and $q_n(\mathbf{x},t)$ distribution in 2-D case are presented in [4-6].

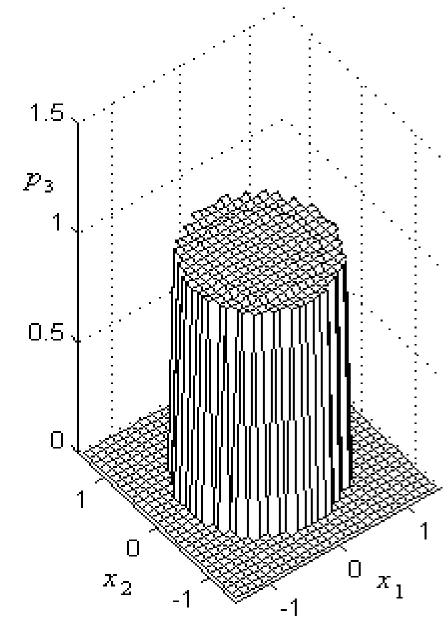

Fig. 2. Distribution of $q_3(\mathbf{x},t)$ at time $t = 4\pi/5\omega$ for $k_1 = 0.45$.

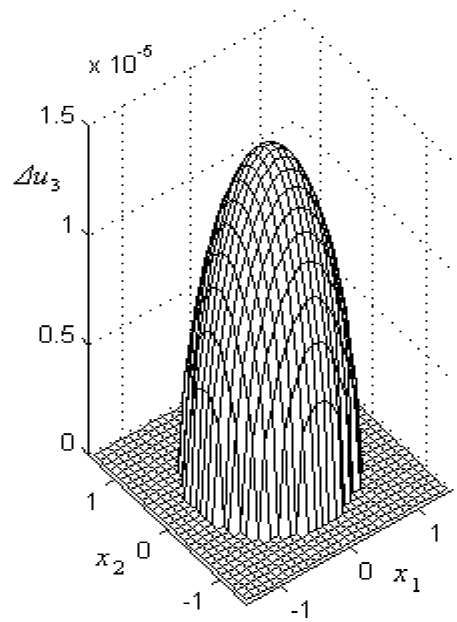

Fig. 3. Distribution of $\Delta u_3(\mathbf{x},t)$ at time $t = 0$ for $k_1 = 0.45$.

The results of the SIF calculations are presented in Fig. 4, where the dimensionless SIF $K_I^{\max}/K_I^{stat}$ is given. Here $K_I^{\max}$ is a maximum value of $K_I(t)$ and K_I^{stat} is a corresponding static value. Curve 1 corresponds to the problem without contact interaction, curve 2 corresponds to the problem with contact interaction of the crack edges. From these graphs it follows that considering of the crack edges contact interaction, the maximal SIF exceeds the corresponding static values by 20%. In the problem without considering the contact interaction, dynamic SIF exceeds static SIF by 55%. However, contact interaction changes the solution both quantitatively and qualitatively. The maximums of curves 1 and 2 in Fig. 4 do not coincide.

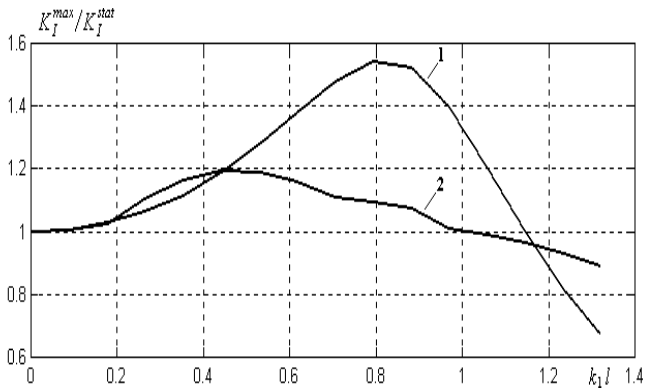

Fig. 4. Dependence of $K_I^{\max} / K_I^{stat}$ on the wave number: 1 - without contact, 2 - with contact.

7 Conclusion

Unilateral elastodynamic contact problems for bodies with cracks was considered here using a variational approach. The results presented here and in previous our publications show that the contact interaction of the crack edges influences the fracture mechanics criterions. This must be taken into account when the strengths of structures are calculated using the fracture mechanics methods.

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