Numerical Simulation of Drift-Diffusion Traffic Flow Model

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Abstract: - In this study, we present a numerical scheme to solve the drift-diffusion traffic flow model under the steady state. The drift-diffusion traffic flow model consists of a continuity equation and a nonlinear Poisson equation. The continuity equation describes the propagation of density along the road, and the Poisson equation describes the interaction among vehicles. The system equations cannot be solved analytically. Therefore, a numerical iterative scheme, which is a finite difference approximation of the model, is presented. A numerical example is employed to explain the model and to show the advantage of the scheme.

Key-Words: - Traffic Flow, Finite Difference Method, Poisson Equation, Continuity Equation, Decoupled Scheme, Scharfetter-Gummel Scheme.

1 Introduction

Traffic congestion generates the interest in traffic flow researches. Traffic flow theory is a new science, which has addressed questions related to understanding traffic processes and to optimizing these processes through proper design and control. There are four main methodologies: car-following models, kinetic models, Boltzmann-like models and cellular automation (CA). Zhang and Owen [1] compared the advantages of macroscopic and microscopic simulators and mentioned that if the purpose of the research focuses on real-time prediction, macroscopic models is preferred.

Lighthill, Whitham [2] and Richards [3] firstly proposed a macroscopic dynamic traffic flow model (LWR model), which is a continuity equation. The related subjects are broadly researched and debated. The assumption of LWR model is that velocity changes instantaneously as density changes. It is certainly not valid in some traffic flow situations. To overcome the steady state assumption of velocity, Payne [4] used a motion equation to obtain time variant speed and proposed a second order model, which was named as PW model. However, this kind of models has a lot of arguments, so families of gas-kinematic models [5-8] are presented. Cho and Lo [9] also presented a dynamic multiclass multilane traffic flow model by a similar deriving procedure. The systematic model includes continuity, motion and variance equations, so as to describe the evolution of the traffic flow. The state of the traffic flow is determined by the vehicular dispersion model [10], which is a nonlinear Poisson equation.

The systematic model describes the traffic situations that the variation of density, velocity and variance are significant. Fortunately, traffic flow situations are not always so complicated. In this study, a simplified model, which discusses time invariant velocity and variance traffic, is presented. Under the situation, the relation among flow, velocity and density can be represented by a drift-diffusion function [9]. Therefore, the model proposed herein is named as the drift-diffusion traffic flow model. The model, which includes a continuity equation and a nonlinear Poisson equation, cannot be solved analytically. Thus, we propose a numerical algorithm to approximate the solution of the model under steady state. Since the systematic partial differential equation consists of two equations, the concept of decoupling scheme is employed. The Poisson equation is solved by the finite difference scheme straight forward, whereas the continuity equation is solved by the Schafetter-Gummel scheme. After the derivation of the numerical algorithm, a numerical example is used to discuss the model and the traffic situation.

The rest of this paper is organized as follows. Section 2 presents the drift-diffusion traffic flow model and its physical meanings. Section 3 introduces the numerical scheme, which is based on the finite difference method. In section 4, a numerical example is employed to discuss the model. The efficiency and convergence are also demonstrated by the simulation results. After that, the paper concludes with some perspectives in section 5.

2 Model Description

Since a complete dynamic system includes transient equations and state equations, the model proposed in this study also follows the same viewpoint. A continuity equation is proposed to describe the propagation of the density, whereas a dispersion model is proposed as the state equation. Derivation and the physical meanings are shown in brief in the following paragraphs.

The first equation mentioned herein is the the continuity equation [1, 2], which is represented as

$$\partial k/\partial t + \nabla \cdot q = 0\,, \tag{1}$$

where k denotes density and q denotes flow. The continuity equation is derived from the conservation of vehicle numbers and describes the transient of density. Furthermore, the continuity equation can also be derived from multiplying Boltzmann equation by the zeroth moment function [6-9]. The equation is a simple but sufficient traffic theory if velocity and variance are time invariant.

The second equation is the dispersion model [10], which is a nonlinear Poisson equation. The model is derived from the car-following theory [11-12] and the following assumptions. The first one is that each vehicle has its own field and vehicles exclude each other by their own field. The second one is that the traffic field satisfies the inverse-square law. It means the influence of other vehicles is larger when the spacing is smaller. For the sake of safety, one vehicle on a road adjusts its velocity and spacing according to the relative position between other vehicles so as to avoid the accident; that is, density is distributed by the traffic field. The traffic field acting on vehicle 0 is represented as

$$\mathbf{E} = \sum_{i}^{N} \left(e_i \mathbf{X}_i / \varepsilon_i \|\mathbf{X}_i\|^3 \right), \tag{2}$$

where N is the number of vehicles that may interact with vehicle 0, $\mathbf{X}_i$ denotes the spacing from vehicle i to vehicle 0. In the continuous space, equation (2) can be represented as

$$\mathbf{E} = \frac{e}{\varepsilon} \int_{\Omega} \left((k - k_s) / \|\mathbf{X}\|^2 \right) d\Omega\,, \tag{3}$$

where Ω is the road section, e denotes the passenger car equivalent and ε denotes the interacting parameter. k is the actual density and k_s is the passable density under the given condition. Then, a potential function ϕ exists by the potential theory. The potential function ϕ satisfies $\mathbf{E} = -\nabla_x \phi$. Thus, the magnitude of traffic field is illustrated as

$$div\mathbf{E} = -\Delta\phi = e(k - k_s)/\varepsilon + K_a\,, \tag{4}$$

where $div\mathbf{E}$ denotes the magnitude of traffic field, $K_a = K_a(x)$, which depends on the position x, is the adjust term of the road condition if the road condition is ideal $K_a = 0$. Equation (4) only describe that the traffic field (or traffic potential) depends on density.

Because density is distributed by the traffic field, a relation between density and traffic potential is needed. Here, we assume that the density will tend toward its equilibrium distribution, which is the most possible microscopic state under a specific macroscopic situation. Hence, the equilibrium distribution is derived from a mathematical programming, whose objective is finding out the most possible microscopic state under the specific macroscopic phenomena given by the constraints. The results is

$$k = K_0 \exp\left(e(\phi - \psi)/\Theta_e \right), \tag{5}$$

where K_0 is the essential density, ψ is the potential barrier and Θ_e is the equilibrium velocity variance. Equation (5) describes that as traffic potential (the magnitude of traffic field) increases, density decreases, which implies that vehicles with less interaction with others can spread out easily but induces unstable traffic flow at the same time. In the real traffic condition, the density does not spread out immediately as the traffic potential increases. There exists a threshold ψ . When the velocity variance is larger than ψ , the density will be less than the essential density; otherwise the density will be larger than the essential density. The statement means that if $\phi - \psi > 0$, vehicles can move away freely and the density in the interval will be less than the essential density. On the other hand, if $\phi - \psi < 0$, vehicles are trapped in the platoon and the density will be larger than the essential density. There are two more points can be inferred from the equation. The first one is as the equilibrium velocity variance increases, the variation of density increases, which means the traffic is sensitive. The second one is as the potential barrier is low, the density is small; that is, drivers are aggressive. By coupling equations (4) and (5), the dispersion model is obtained.

At last, the explicit form of the fundamental diagram is discussed. Under steady state and homogeneous assumption of velocity and variance, the relation among flow, density and velocity (u) can be obtained from the motion equation [9] can be simplified as

$$q = ku = -\mu\left[\nabla_x \cdot (k\Theta_e)\right] + ek\mu\mathbf{E} = -ek\mu\nabla_x\phi - \nu \cdot \nabla_x k \tag{6}$$

where μ denotes the mobility of vehicles, which is a constant. $\nu = \mu\mathbf{\Theta}_e/e$ denotes the diffusion coefficient.

ν is a constant, too. Equations (1), (4)~(6) are the equations of our model. From equation (6), flow can be described by the combination of drift and diffusion effect. Thus, the model is named as the drift-diffusion traffic flow model. The drift-diffusion traffic flow model describes the evolution of density by the continuity equation and evaluates the interaction (traffic potential) among vehicles by the dispersion equation. Also, the interaction distributes density by the dispersion equation and influences the propagation of density. By observing the equations, all variables can be determined by solving the system. Thus, the drift-diffusion traffic flow model is a self-consistent one.

In this study, we discuss the model under the steady state; i.e., $\partial k/\partial t = 0$ in equation (1). We can rewrite the equations as follows.

$$\nabla_{\mathbf{x}} \cdot \left(-ek\mu\nabla_x\phi - \nu \cdot \nabla_x k\right) = 0, \tag{7}$$

$$-\Delta\phi = \frac{e}{\varepsilon}\left[K_0 \exp\left(-\frac{e\phi - e\psi}{\Theta_e}\right) - k_s\right] + K_a, \tag{8}$$

where equation (7) is the steady state continuity equation and equation (8) is the dispersion equation. After showing the model, the numerical algorithm is illustrated in the following section.

3 Numerical Scheme

Because there are two equations in the model, we present a decoupling algorithm first in this section. That is, the Poisson equation and the continuity equation are solved separately. After the both equations converge, the convergence of the whole system is examined. If the system converges, then the algorithm stops to compute the flow and velocity. The decoupled scheme is illustrated in section 3.1 and the discretized scheme of equations (7) and (8) are illustrated in section 3.2.

3.1 Decoupling Scheme

Decoupling scheme is one of efficient solution methods in simulation [13]. The method decouples the system as two independent partial differential equations and then solves each partial differential equation iteratively. The basic idea of decoupling method is that the decoupled equations are solved sequentially, which is illustrated in figure 1. If we denote ϕ^n as the approximated potential of the nth outer loop iteration. In addition, k^n denotes the nth outer loop iteration.

Solving procedure of the decoupling scheme is shown as follows. Firstly, Poisson's equation is solved for ϕ^n by given the previous states k^{n-1}. Secondly, the continuity equation is solved for k^n by given ϕ^n. Then, the convergent criterion of outer loop is examined. Sup norm error is the chosen convergent criterion in this study. Each decoupled partial differential equation is solved by the inner iterative loop. The scheme of inner loop is presented in the next section.

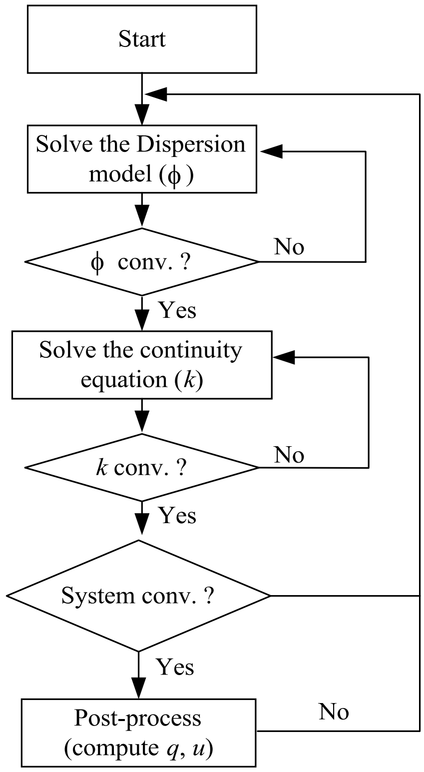

Fig. 1. A flowchart for decoupling scheme in traffic flow simulation.

3.2 Discretization of the Model

In this section, application of finite difference approximation to nonlinear Poisson equation (dispersion model) is presented, whereas the continuity equation is discretized by the Scarfetter-Gummel scheme [12, 13]. Firstly, the scheme of the nonlinear Poisson equation is derived as follows.

From the fundamental theorem of integral calculus, we have

$$\int div\nabla \cdot \phi dx = \int\left[\frac{e}{\varepsilon}\exp\left(\frac{e\psi - e\phi}{\Theta_e}\right) - \frac{e}{\varepsilon}k_s + K_a\right]dx \cdot \tag{9}$$

By central difference method, we can derive a direct computation equation (10).

$$\left(\frac{\phi_{i+1}-\phi_i}{h_i}\right)+\left(\frac{\phi_{i-1}-\phi_i}{h_{i-1}}\right) = \left(\frac{h_i+h_{i-1}}{2}\right)\left[\frac{e}{\varepsilon}\exp\left(\frac{e\psi_i-e\phi_i}{\Theta_e}\right)-\frac{e}{\varepsilon}k_{si}+K_{ai}\right], \tag{10}$$

where h_i is the length of the interval *i*.

The discretization of the continuity equation is much more crucial. First we write the approximation of flow as

$$q(x\in[x_i,x_{i+1}])=q|_{i+1/2}+\left(x-x_i-\frac{h_i}{2}\right)\cdot\frac{\partial}{\partial x}q|_{i+1/2} + O(h^2)\frac{\partial^2}{\partial x^2}q|_{i+1/2}. \tag{11}$$

We obtain the approximation by ignoring the $O(h^2)$ terms. For the interval $[x_i,x_{i+1}]$ we have

$$\mu k\frac{\partial\phi}{\partial x}-\nu\frac{\partial k}{\partial x}=q|_{i+1/2}+\left(x-x_i-\frac{h_i}{2}\right)\cdot\frac{\partial}{\partial x}q|_{i+1/2}. \tag{12}$$

This equation is solved to determine the variation of the density along the path $[x_i,x_{i+1}]$. We have to assume that the partial derivative of the traffic potential is constant on the path under consideration, which is the assumption we have already invoked for the Poisson equation. Therefore, equation (12) can be considered as a first order ordinary differential equation of density *k*. The solution is

$$k(x\in[x_i,x_{i+1}])=C\exp\left(\frac{-e\phi(x)}{\Theta_e}\right)+h_i\frac{q|_{i+1/2}}{\mu}\frac{1-\exp\left(\frac{-e\phi(x)}{\Theta_e}\right)}{e(\phi_{i+1}-\phi_i)} + h_i^2\frac{\left(\frac{\Theta_e}{\phi_{i+1}-\phi}\right)\left(1-\exp\left(\frac{-e\phi(x)}{\Theta_e}\right)\right)}{e^2(\phi_{i+1}-\phi_i)}\frac{\frac{\partial}{\partial x}q|_{i+1/2}}{\mu}. \tag{13}$$

Assuming that $|\phi_{i+1}-\phi_i|=O(h)$, shows that the last term in equation (13) is $O(h^3)$. By ignoring this term, we obtain

$$k(x\in[x_i,x_{i+1}])=(1-g_i(k,\phi))k_i+g_i(k,\phi)k_{i+1}. \tag{14}$$

with

$$g_i(k,\phi)=\frac{1-\exp\left(\frac{e\phi_i-e\phi_{i+1}}{\Theta_e}\frac{x-x_i}{h_i}\right)}{1-\exp\left(\frac{e\phi_i-e\phi_{i+1}}{\Theta_e}\right)}. \tag{15}$$

If $\phi_{i+1}=\phi_i$, the growth function (14) degenerates to a linear function.

$$g_i(k,\phi)=\frac{x-x_i}{h_i}. \tag{16}$$

Therefore, the discretized scheme of continuity equation is represented as

$$\begin{aligned}
&\frac{(q_{i+1/2}-q_{i-1/2})}{(h_i+h_{i-1})/2} \\
&=\frac{2}{(h_i+h_{i-1})}\Bigg[\mu e\left(\left(\frac{k_{i+1}}{1+\exp\left(\frac{\phi_i-\phi_{i+1}}{2(\Theta_e/e)}\right)}\right)+\left(\frac{k_i}{1+\exp\left(\frac{\phi_{i+1}-\phi_i}{2(\Theta_e/e)}\right)}\right)\right)\frac{\phi_{i+1}-\phi_i}{h_i} \\
&+\mu e\left(\left(\frac{k_{i-1}}{1+\exp\left(\frac{\phi_i-\phi_{i+1}}{2(\Theta_e/e)}\right)}\right)+\left(\frac{k_i}{1+\exp\left(\frac{\phi_{i-1}-\phi_i}{2(\Theta_e/e)}\right)}\right)\right)\frac{\phi_{i-1}-\phi_i}{h_{i-1}} \\
&-\nu\frac{\frac{\phi_{i+1}-\phi_i}{\Theta_e/e}}{\exp\left(\frac{\phi_i-\phi_{i+1}}{2(\Theta_e/e)}\right)-\exp\left(\frac{\phi_{i+1}-\phi_i}{2(\Theta_e/e)}\right)}\frac{k_{i+1}-k_i}{h_i} \\
&-\nu\frac{\frac{\phi_{i-1}-\phi_i}{\Theta_e/e}}{\exp\left(\frac{\phi_{i-1}-\phi_i}{2(\Theta_e/e)}\right)-\exp\left(\frac{\phi_i-\phi_{i-1}}{2(\Theta_e/e)}\right)}\frac{k_{i-1}-k_i}{h_{i-1}}\Bigg] \\
&=0
\end{aligned} \tag{17}$$

Since density is function of potential, the approximation k_i is

$$k_i=K_{0i}\exp\left(\frac{\psi_i-\phi_i}{\Theta_e/e}\right) \tag{18}$$

By substituting equation (18) for k_i in equation (17), we can derive the iterative scheme. The derivation of the scheme is followed the idea proposed by Scharfetter and Gummel [12]. In this scheme, a finite difference approximation with exponentially fit the dependent variables is employed for the continuity equation. Because traffic potential is expected to change smoothly, whereas density may change dramatically.

4 Numerical Example

A complete problem of the model should include a set of boundary condition. The boundary conditions of potential is transformed from the boundary conditions of density. Thus, boundary conditions of density in traffic flow problems are discussed first. If the traffic condition on a boundary can be described by a deterministic function, a Dirichlet condition is chosen, such as deterministic inflow or outflow and no entrance and exit on the roadsides. If we only have the changing rate on a boundary, a Neumann condition is chosen. The situation takes place at the ramp, the signal intersection and the toll collection station where the service rate is easily obtained. Mostly, a boundary is mixed by both Dirichlet and Neumann condition, which is a Robin condition. A traffic flow problem often involves two or three types of boundary conditions, which is a mixed boundary condition problem. As a boundary condition of density is formulated, with proper assumption of driving behavior and interaction a boundary condition of potential is obtained. The numerical example in this study considers a basic section of highway, which is behind an intersection. The research domain is a single lane section with 1 km in length. Under the uncongested traffic, the boundary conditions are given as follows:

$$\begin{cases} \frac{1}{2}\left(\phi(0)-\frac{\partial\phi}{\partial\mathbf{n}}\right)=3-\frac{\Theta_e}{e}\ln\left(\frac{k_s(0)}{K_0}\right) \\ \phi(100)=-\frac{\Theta_e}{e}\times\ln\left(\frac{k_s(100)}{K_0}\right) \\ k(0)-\frac{1}{3}\frac{\partial k}{\partial\mathbf{n}}=50 \\ k(100)=30 \end{cases} \tag{20}$$

where the passenger car equivalent $e=1$, the equilibrium velocity variance $\Theta_e=16$ km^{2}phr^2 and the $K_0=100$ vehpkmpl. Assume the interacting parameter $\varepsilon=3{,}600$ and the domain is ideal in geometry then $K_a=0$. The first condition in equation (20) means that vehicles, which is going to enter the interval is resisted by the existent vehicles. In addition, vehicles in the interval are forced to drive forward by an imaginary external potential. The second condition means the outflow depends on the existent density. The third one means that the density of inflow is determined by the inflow density from the front interval. The last condition gives that the outflow density is uniform. Through the boundary conditions, traffic flow is forced to move forward and the mobility of the vehicles is blocked by the density (platoon of vehicles). The unstrained density is assumed to describe that the front part of the research interval is disturbed by the inflow vehicles form different direction. Gradually, it gets back to the average value. The unrestrained density k_s is assumed to be

$$k_s=\begin{cases} 30+0.125*x, & x\in[0,80] \\ 40, & x\in[80,100] \end{cases}. \tag{21}$$

The numerical results are illustrated in figures 2 to 5. Traffic potential decays, whereas density increases in the front part of the interval and then decreases. It means that the interaction among vehicles in the front part is large and forms a platoon. This results in the increase of the density and decrease of the velocity, which is shown in figure 5. Velocity, which is obtained from *u=q/k*, decreases firstly and then increases. Flow rate is illustrated in figure 4. It is almost a constant. The result is reasonable, since the model solved herein is under steady state.

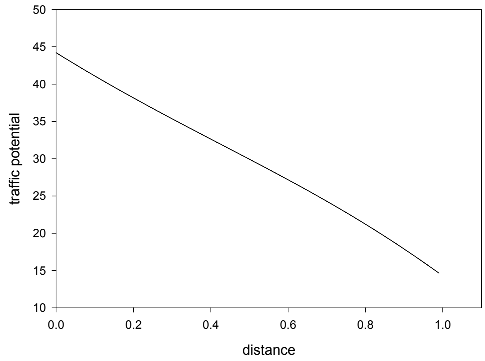

Fig. 2. The plots of potential in numerical example.

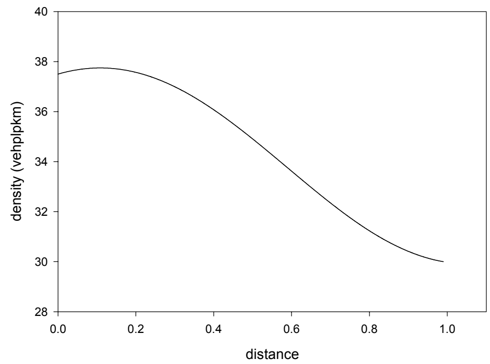

Fig. 3. The plots of density in numerical example.

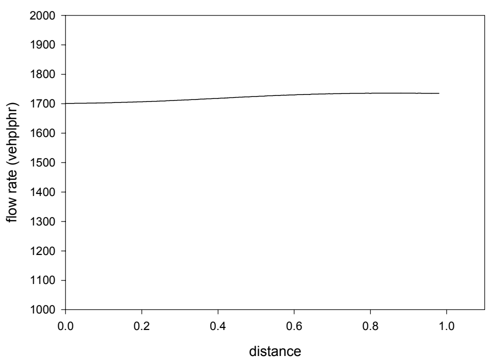

Fig. 4 The plots of flow rate in numerical example.

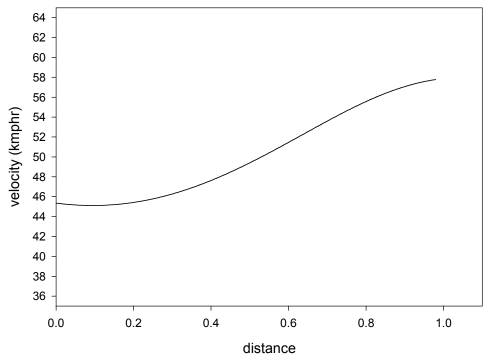

Fig. 5 The plots of velocity in numerical example.

5 Conclusion

Traffic flow theory is the fundamental research of traffic science. In this study, a drift-diffusion traffic flow model is presented. The model is also a self-consistent one, which describes the evolution of density by the continuity equation and determined the interaction among vehicles by the density. A decoupling scheme is also suggested to solve the system under the steady state. Furthermore, a numerical example of uncongested highway is employed to explain the traffic condition.

There are still further works left for further researches, such as numerical scheme and simulations of time-dependent problem, scenarios of different traffic situation, the application of traffic control, extension of multilane traffic model and so on. In addition, as providing the real-time information and prediction become more and more important today, the efficiency and accuracy of numerical simulation are further important research topics.

Acknowledgments

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