New Trajectory Tracing Method for Express Study of the Evolution of Non-linear Systems

P.P. HORLEY[a,b], P.YU. VOROBIEV[c], P.M. GORLEY[b]
YU.V. VOROBIEV[a] and J. GONZÁLEZ-HERNÁNDEZ[a]

[a] CINVESTAV-IPN, Libramiento Norponiente #2000, Juriquilla, 76230 Querétaro, MÉXICO

[b] Chernivtsi National University, 2 Kotsyubynsky str., 58012 Chernivtsi, UKRAINE

[c] Facultad de Ingeniería, Universidad Autónoma de Querétaro, 76010 Querétaro, MÉXICO

Abstract: - New numerical trajectory tracing method is proposed to investigate the evolution of non-linear and chaotic system. The method is very simple and easy for computer implementation, and also is very fast in calculations. This method allows us to investigate the topological changes of phase portraits even in chaotic states and to find quickly and precisely the bifurcation points of the system. The method was successfully tested with classic Rössler and Lorentz strange attractors.

Key-Words: - non-linear systems, chaos, self-organization, phase portrait, topology, bifurcation, system evolution

1 Introduction

During the past decades significant attention is being paid to the investigation of non-equilibrium and non-linear systems, as well as the dynamics of the self-organization processes and chaos in them. Self-organizing processes appear in physical, chemical, biological, social and other systems, which contain many particles, exchange mass, energy or information with the environment and are far from the equilibrium state. Analytical investigation of such complicated processes is impossible; thus the main part belongs to numerical methods and their computer implementation.

2 Trajectory tracing method

The set of the phase points, characterizing the state of the system under the given combination of control parameter values, could be obtained either by numerical solution of non-linear differential equations that describes the system or by experimentally measured data. To determine the influence of the control parameters on the system investigated one need to process obtained phase portrait data with different methods [1-2], such as bifurcation diagrams, Poincaré section, Lyapunov exponents, Hausdorff dimension, correlation dimension and others to find the level of chaotization and stability of the system, analyze its behavior near bifurcation points, etc. But all the named standard methods have rather complicated algorithms and therefore need significant calculation time.

We are proposing a new method to solve this problem and to allow rapid and accurate investigation of the evolution of the system depending on the control parameter changes. The idea of the method is to perform tracing of phase point trajectory. Let us consider the latter – the phase portrait – as the set of S vectors. The topology of the phase portrait is being defined by the order of the directions of these vectors, as well as sequence of their lengths. If we have n-dimensional phase portrait, one can consequently define the cosine of the angle θ_i between the two consecutive vectors V_i and V_{i+1} using n-dimensional scalar product:

$$\cos\theta_i = \frac{\sum_{j=1}^{n} [V_i]_j [V_{i+1}]_j}{\sum_{i=1}^{n} [V_i]_j^2 \sum_{i=1}^{n} [V_{i+1}]_j^2}, \qquad (1)$$

where j means unit vector of n-dimensional Cartesian system. In this way one can unambiguously represent the phase portrait by a single trajectory tracing curve

(TTC) – the sequence of $\cos\theta_i$. In the case when phase point trajectory is a limit cycle, i.e. running angle between consecutive vectors changes periodically with the same period, the trajectory tracing curve will oscillate with period T, where T is the amount of iterations/steps between two points with the same topological position. If period doubling bifurcation occurs, one can note the doubling of the period T of the tracing curve, etc. Therefore trajectory tracing method could be used to observe the topology changes with changes of control parameter value, providing us with the information about ordered and chaotic states of the system as well as the type of transitions between them.

3 Results and discussion

Trajectory tracing method was checked for classical chaotic systems:

Rössler attractor [3] (Fig. 1)

$$\begin{cases} dx/dt = -y - z \\ dy/dt = x + \alpha y \\ dz/dt = \beta + z(x - \mu) \end{cases}, \quad (2)$$

Lorentz attractor [4] (Fig. 2)

$$\begin{cases} dx/dt = \sigma(y - x) \\ dy/dt = -xz + rx - y \\ dz/dt = xy - bz \end{cases}. \quad (3)$$

In the figures we have presented the plot of so called trace value, i.e. renormalized cosine values as $\lg[(10^5 - 10)\cdot(1 - \cos\theta_i) + 10]$, because the original cosine values are very small and moreover, have different order of magnitude, so we need to make this or similar transformation to emphasize the details of the curves. Cosine is being substracted from the unity to avoid its negative values. Tracing depth represents the phase point number in the phase portrait and in our case corresponds to different values of independent variable *t*. To obtain clear patterns it is necessary to omit some initial data points, when phase trajectory tends to attractor but does not lie on it. This initial number is referred as zero tracing depth in our figures. The increase of tracing depth corresponds to increase of independent variable *t* with integration step *h*, which is given on the figure captions. Let us note that *h* defines the smoothness of the phase trajectory and the trajectory tracing curve.

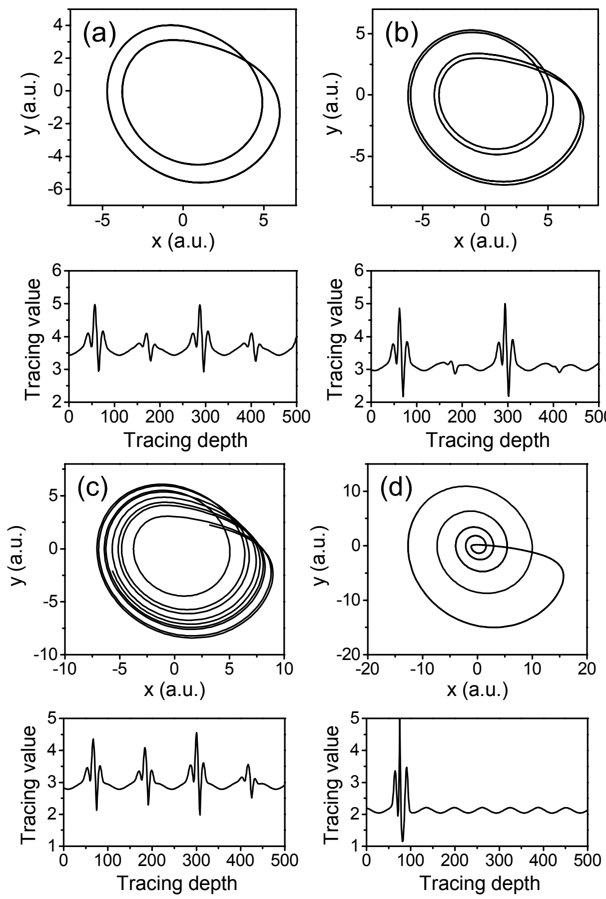

FIG. 1. Trajectory tracing curves for Rössler attractor: a) μ = 3.000, b) μ = 3.875, c) μ = 4.438, d) μ = 8.062. Other control parameters: α = 0.2, β = 0.2. Integration step h = 0.05.

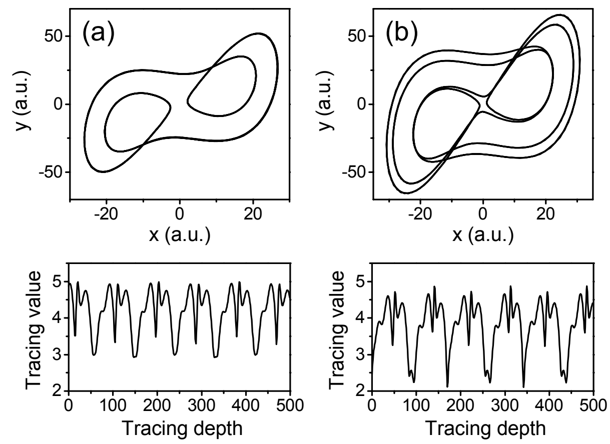

FIG.2. Trajectory tracing curves for Lorenz attractor: a) r = 98.125, b) r = 113.75. Other control parameters: σ = 6, b = 2. Integration step h = 0.01.

It is clearly seen that the two attractors have different distinctive kinds of trajectory tracing curves, which are specific for the given system. Ordering leads to simple TTCs while chaotization of the system adds to it new peaks, but the overall picture represents the same system-specific pattern. For example, sharp peaks for the Rössler system correspond to the upper loop of the phase portrait, that does not lie in (x,y) plane. Therefore trajectory tracing curve gives us the information about the system topology, namely about the order of movement of phase point within the attractor. Period doubling bifurcations clearly mark themselves by period doubling of the TTC (Fig. 1 (a,b), Fig. 2 (a,b)), allowing us to define the period by analyzing just one dependence instead of n-dimensional phase portrait.

To investigate the evolution of the phase portrait of non-linear system, one should build the set of trajectory-tracing curves for different values of control parameter. The calculated data are presented in the Fig. 3. The tracing values are represented by color – darker areas correspond to smaller values, lighter – to greater ones. One can see that the characteristic TTC pattern is observable in all the region of control parameter changes. From the figure it follows that Rössler attractor (Fig. 3a) is more ordered system – the TTC pattern is simple and undergoes the multiplication of the peaks. The bifurcation points correspond to the values of control parameter, when new peaks arise on TTC. Chaotic states appear from the ordering ones by successive period doubling bifurcations – hence the system obeys the Landau-Hopf evolution model. Lorenz attractor (Fig. 3b) shows chaotic behavior, as his TTC are shifted randomly. These changes could be explained by the changes of topology of the phase portrait – phase point changes the order of the movement over two petals of attractor. Ordered states with simple TTC appears seldom and are rapidly followed by chaotic ones. That is characteristic to Pomo and Manewill model of intermittence of ordered and chaotic states.

Therefore the data obtained with trajectory tracing allow us to investigate qualitatively the evolution of the states of two classical strange attractors as well as to define the bifurcation points.

4 Conclusion

Proposed trajectory tracing method is the powerful and simple way to obtain information about phase portrait topology of non-linear system and to investigate its evolution with changes of control parameter. Calculations performed for classic strange attractors proved its accuracy, allowing us to use it for different problems of non-linear, chaotic and self-organization systems.

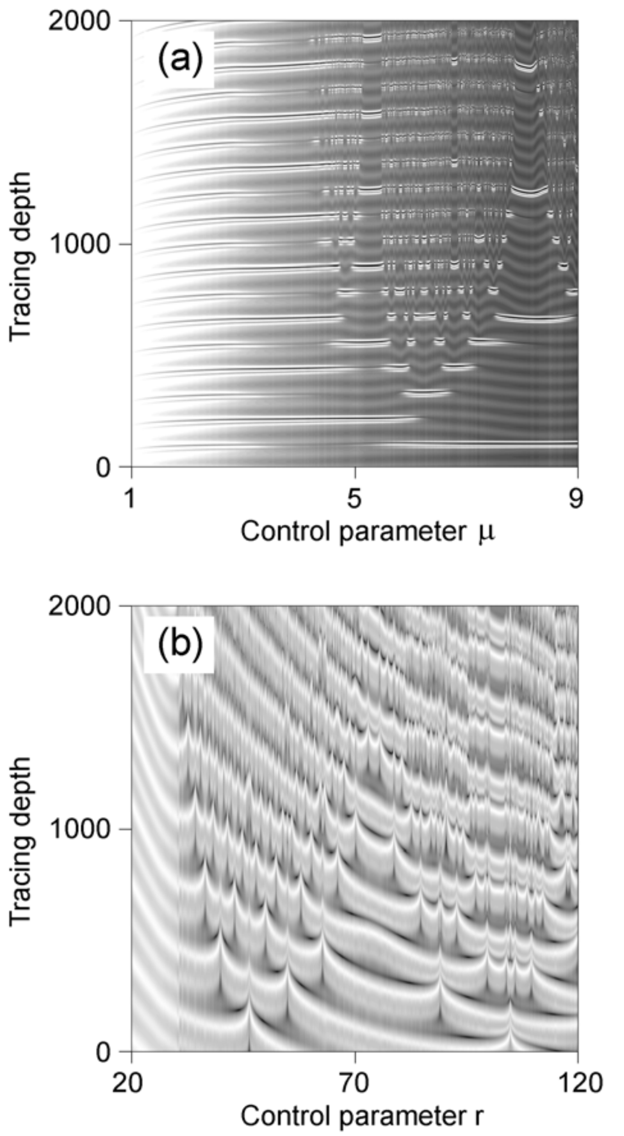

FIG. 3. Evolution of trajectory tracing curves for a) Rössler attractor and b) Lorentz attractor depending on the changes of their control parameters μ and r. Values of other parameters are given in captions for figures 1 and 2.

References:

[1] G. Nicolis and I. Prigogine, *Self-Organization in Non-Equilibrium Systems: From Dissipative Structures to Order through fluctuations*, J. Wiley & Sons, 1977

[2] A.J. Lichtenberg M.A. Liberman, *Regular and stochastic motion*, Springer-Verlag, 1983

[3] O.E. Rössler, An Equation for Continuous Chaos, *Physics Letters A*, Vol.57, 1976, pp. 397-398.

[4] E.N. Lorenz, Deterministic Nonperiodic Flow, *Journal of the Atmospheric Sciences*, Vol.20, 1963, pp. 130-141.