Synchronization of two hyperchaotic Rössler systems: Model-matching approach[1]

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Abstract: In this paper, a study of synchronization problem of hyperchaotic discrete-time systems is presented. In particular, we use a model-matching approach to synchronize two unidirectionally coupled Rössler systems.

Key-Words: Synchronization, hyperchaotic system, discrete-time nonlinear system, model-matching problem, Rössler system.

1 Introduction

Synchronization of chaotic systems has received a lot of attention in last decade, this interest increase by practical applications in different fields particularly in private communication. Since the work of Pecora and Carroll [1] different approaches are being currently proposed and pursued [2]-[11].

The objective of this paper is to present another scheme to synchronizing chaotic systems using the model-matching approach. In this work we show that the synchronization of two hyperchaotic Rössler systems is possible from this point of view.

This paper is organized as follow. In Section 2, we present the problem description. In Section 3, we describe the nonlinear model-matching approach. In Section 4, we present the synchronization of two hyperchaotic Rössler systems. Finally, in Section 5, we give some concluding remarks.

2 Problem Formulation

Consider a discrete-time nonlinear system described by

$$\begin{aligned} x(k+1) &= f(x(k), u(k)), \\ y(k) &= h(x(k)), \end{aligned} \tag{1}$$

where the state $x \in X$, an open set in $\Re^n$, the input u is inside an open set U in $\Re^m$, and the output y belongs to an open set Y in $\Re^m$. The mapping $f : X \times U \to X$ and $h : X \to Y$ are supposed to be analytic. In addition, we consider another discrete-time nonlinear system described by

$$\begin{aligned} x_M(k+1) &= f_M(x_M(k), u_M(k)), \\ y_M(k) &= h_M(x_M(k)), \end{aligned} \tag{2}$$

where the state $x_M \in X_M$ (an open set in $\Re^{n_M}$), the input $u_M \in U_M$ (an open set in $\Re^m$), and the output y_M belong to an open set Y_M in $\Re^m$. Also, the mapping $f_M : X_M \times U_M \to X_M$ and $h_M : X_M \to Y_M$ are supposed to be analytic.

We said the system (1) synchronize with system (2) if

$$\| y_M(k) - y(k) \| \to 0, \qquad k \to \infty. \tag{3}$$

note that in this case we consider only output synchronization [12]. In the next section we describe the scheme to solve (3).

3 Nonlinear Model-Matching Problem

Consider a system called the *plant P* described by (1) and a system called the *model M* described by (2). The compensator utilized to control the plant is a discrete-time nonlinear system described by

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$$C:\begin{cases} x_C(k+1) = f_C(x_C(k), x(k), u_M(k)), \\ u(k) = h_C(x_C(k), x(k), u_M(k)), \end{cases} \tag{4}$$

with state $x_C \in X_C$ (an open set in $\Re^{\nu}$, input x and u_M and output u. Here f_C and h_C are analytic mappings on a suitable open and dense subset of $X_C \times X \times U_M$. The compensated plant (1)-(4) defines a new control system, with input u_M, output $y_{P\circ C}$, and state $\hat{x} = (x, x_C)$, described by

$$P \circ C:\begin{cases} \hat{x}(k+1) = \hat{f}(\hat{x}(k), u_M(k)), \\ y_{P\circ C}(k) = \hat{h}(\hat{x}(k)), \end{cases} \tag{5}$$

where

$$\hat{f}(\hat{x}, u_M) = \begin{pmatrix} f(x, h_C(x_C, x, u_M)) \\ f_C(x_C, u_M) \end{pmatrix}, \quad \hat{h}(\hat{x}) = h(x).$$

We assume that plant (1) evolves in a neighbourhood of an equilibrium point x^0; that is, around $(x^0, u^0) \in X \times U$ such that $f(x^0, u^0) = x^0$, with $\{u(k) = u^0 : k \geq 0\}$ being a (constant) input sequence. For this sequence there exists another (constant) output sequence $\{y(k) = h(x^0) = y^0 : k \geq 0\}$. In the same way, let the equilibrium point of (2) be denoted by $(x_M^0, u_M^0) \in X_M \times U_M$. We are interested in finding a compensator C for the plant P which, irrespectively of the initial states of P and M, makes the output *y(k)* of P asymptotically converge to the output $y_M(k)$ produced by M under an arbitrary input $u_M(k)$ to solve synchronization problem described in section 2. In this case we used the nonlinear model-matching approach [17], [19] to obtain (3). The discrete nonlinear model-matching problem addressed here is defined as follows.

Definition 1 *(Discrete-time asymptotic nonlinear model-matching problem, DAMMP) Given the plant P and the model M around their respective equilibrium points* (x^0, u^0) *and* (x_M^0, u_M^0), *and a point* $(x(0), x_M(0)) \in X^0 \times X_M^0 \subset X \times X_M$ *find an integer ν and a compensator C with initial condition* $x_C(0)$, *such that the output of the compensated plant* $y_{P\circ C}$ *converges asymptotically to the output* $y_M(k)$ *produced by any input* $u_M(k)$ *to the model M. This means finding two mappings* f_C *and* h_C *such that the compensated plant satisfies the propperty given before in some neighbourhoods* V_1 *of* (x_C^0, x^0, u_M^0) *in* $X_C \times X \times U_M$ *and* V_2 *of* u^0 *in U.*

A way to solve the above problem is to define an output error $y_E(k) = y(k) - y_M(k)$, and to choose a control $u(k)$ such that $y_E(k)$ is decoupled from the model input $u_M(k)$ for all $k \geq 0$, and converges asymptotically to zero. The first point is equivalent to transforming the *DAMMP* to a *disturbance-decoupling problem with disturbance measurements* of an auxiliary system [17], [19]. Such an approach also allows us to treat the second point in such a way that the output error depends only on the initial conditions $x(0)$ and $x_M(0)$, and not on the model input $u_M(k)$. To this end, let us define the *auxiliary system*

$$E:\begin{cases} x_E(k+1) = f_E(x_E(k), u_E(k), w_E(k)), \\ y_E(k) = h_E(x_E(k)), \end{cases} \tag{6}$$

with state $x_E = (x, x_M)^T$, input vectors $u_E = u$ and $w_E = u_M$, and

$$f_E(x_E, u_E, w_E) = \begin{pmatrix} f(x, u) \\ f_M(x_M, u_M) \end{pmatrix},$$
$$h_E = h(x) - h_M(x_M).$$

Note that w_E is considered as a disturbance signal acting on the auxiliary system. Given this system, together with an equilibrium point $x_E^0 = (x^0, x_M^0)$ it is known [13]-[15] that, if the disturbance-decoupling problem with measurement disturbance w_E associated with the system has a solution on Ω_0^E, an open and dense subset of $X \times X_M \times U \times U_M$, defined around the equilibrium point (x^0, x_M^0, u^0, u_M^0), then there exists an analytic mapping y^E defined on Ω_0^E with the property that the *control law.*

$$u(k) = \gamma^E(x_E(k), w_E(k)) = \gamma^E(x_E(k), u_M(k)) \tag{7}$$

decouples the output $y_E(k)$ of the closed-loop system (6)-(7) from the disturbance $w_E(k)$ for every initial state $x_E(k)$ in an open and dense subset of $X \times X_M$ contained in Ω_0^E.

In the sequel the DAMMP will be treated in terms of a *relative degree* associated with each output components y_i and y_{M_i}. Thus the following definitions are introduced. Let f_0, f_{M_0}, and f_{E_0} be the undriven state dynamics $f(\cdot,0)$, $f_M(\cdot,0)$, and $f_E(\cdot,0,0)$, respectively, and f_0^j, $f_{M_0}^j$, and $f_{E_0}^j$ the *j*-times iterated compositions of f_0, f_{M_0}, and f_{E_0} with $f_0^0(x)=x$, $f_{M_0}^0(x_M)=x_M$, and $f_{E_0}^0(x_E)=x_E$.

Definition 2 [16] *The output y_i of the plant (1) is said to have a relative degree d_i in an open and dense subset O_i of $X\times U$ containing the equilibrium point (x^0,u^0), if*

$$\frac{\partial}{\partial u_j}\left[h_i\circ f_0^l(f(x,u))\right]\equiv 0 \tag{8}$$

for all $0\le l\le d_i-1$, for all $1\le j\le m$, for all $(x,u)\in O_i$, and

$$\frac{\partial}{\partial u_j}\left[h_i\circ f_0^{d_i}(f(x,u))\right]\neq 0 \tag{9}$$

for some $j\in\{1,\ldots,m\}$ and for all $(x,u)\in O_i$.

A similar definition is given for the relative degree of the model (2), d_{M_i}, in an open and dense subset O_{M_i}, of $X_M\times U_M$ containing the equilibrium point (x_M^0,u_M^0).

We define the *input-output decoupling matrix* of the plant *P* as the $m\times m$ matrix

$$A(x,u)=\frac{\partial}{\partial u_j}\left[h_i\circ f_0^{d_i}(f(x,u)):1\le i,j\le m\right], \tag{10}$$

and consider the following assumption.

(A1) For all $1\le i\le m$ and for all $x_E=(x,x_M)\in X\times X_M$ and $u_M\in U_M$,

$$0\in \operatorname{Im}\left\{h_{E_i}\circ f_{E_0}^{d_i}(f_E(x_E,\cdot,u_M))\right\}, \tag{11}$$

where $\operatorname{Im}\{\varphi\}$ denotes the image of φ.

The following theorem gives necessary and sufficient conditions for the local solvability of the DAMMP.

Theorem 1 [17] *Consider the plant (1) and the model (2) around, respectively, their equilibria (x^0,u^0) and (x_M^0,u_M^0). Suppose that the output y_i of the plant and the output y_{M_i} of the model have finite relative degree d_i and d_{M_i}, respectively defined on O_i and O_{M_i}, for i=1,...,m. Assume that assumption A1 holds. Suppose also that the input-output decoupling matrix $A(x,u)$ in (10) is nonsingular for all $(x,u)\in X\times U$. Then the DAMMP is locally solvable on Ω_0^E if, and only if,*

$$d_i\le d_{M_i},\qquad 1\le i\le m \tag{12}$$

Next we show the representation of the auxiliary system (6) feedback by control law (7) in terms of the plant (1) and the model (2) in different coordinate frame.

Suppose that the plant is fully linearizable, i.e. $\sum_{i-1}^m(d_i+1)=n$. The definition of d_i shows that $h_i(x),\ldots,h_i\circ f_0^{d_i}(x)$ are independent functions [20] and can be chosen as new coordinates $\xi(x)=[\xi_1(x),\ldots,\xi_m(x)]^T$ with $\xi_i(x)=[\xi_{i,1}(x),\ldots,\xi_{i,d_i+1}(x)]^T=[h_i(x),\ldots,h_i\circ f_0^{d_i}(x)]^T$ for *i=1,…,m,* defined on the subset $O=O_1\cap\ldots\cap O_m$ locally around x^0, as follows: $\xi_{i,j}(x)=h_i\circ f_0^{j-1_i}(x)$, for all *j=1,…,*$d_i+1$ and for all *i=1,…,m.* Let us consider the auxiliary system (6) and the new coordinates functions

$$(\zeta(x_E),x_M)=\phi(x_E)=\phi(x,x_M), \tag{13}$$

where now $\zeta(x_E)=[\zeta_1(x_E),\ldots,\zeta_m(x_E)]^T$ with $\zeta_i(x_E)=[\zeta_{i,1}(x_E),\ldots,\zeta_{i,d_i+1}(x_E)]^T$ and

$$\zeta_{i,j}(x_E)=h_{E_i}\circ f_{E_0}^{j-1}(x_E)=\xi_{i,j}(x)-h_{M_i}\circ f_{M_0}^{j-1}, \tag{14}$$

for all *j=1,…,*d_i+1, and for all *i=1,…,m*. This is indeed an admissible choice because the Jacobian matrix $\partial[\phi(x_E)]/\partial x_E$ is nonsingular. By choosing $u=\gamma^E(x_E,u_M)$, the representation of the systems (2) and (6) in the new coordinates takes the form

$$\begin{aligned}
&\zeta_{i,j}(k+1)=\zeta_{i,j}(k),\\
&j=1,\ldots,d_i;\quad i=1,\ldots,m,\\
&\zeta_{i,d_i+1}(k+1)=-\alpha_{i,0}\zeta_{i,1}(k)-\ldots\\
&\qquad\alpha_{i,d_i}\zeta_{i,d_i+1}(k),\\
&x_M(k+1)=f_M(x_M(k),u_M(k)),\\
&y_{E_i}(k)=\zeta_{,1}(k),\qquad i=1,\ldots,m.
\end{aligned} \tag{15}$$

From these last expressions we see that the outputs $y_i(k)$ of the plant differ from the outputs $y_{M_i}(k)$ of the model by a signal $y_{E_i}(k)$ obeying the linear difference equation. We can also identify two subsystems in the closed-loop system (15), namely:

1. The subsystem described by the equation
$$x_M(k+1)=f_M(x_M(k),u_M(k)),$$
which represents the dynamics of the model, and
2. The subsystem described by the equations
$$\zeta_i(k+1)=A_i^*\zeta_i(k),\qquad i=1,\ldots,m,$$

with

$$A_i^* = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_{i,0} & -\alpha_{i,1} & -\alpha_{i,2} & \cdots & -\alpha_{i,d_i} \end{bmatrix}$$

which represents the dynamics of the signals $y_{E_i}(k)$.

Since the dynamics of M can be made stable by assumption, if we choose the control law such that all the eigenvalues of the matrix A_i^* have magnitude strictly less than 1, then the closed-loop system will be exponentially stable and condition (3) holds, solving then synchronization problem defined in sec. 2.

4 Synchronization of Rössler System

In this section we will illustrate the theoretic set up of section 3 by means of a illustrative example.

Consider the discrete time nonlinear system described by

$$\begin{aligned} x_1(k+1) &= \alpha x_1(k)(1-x_1(k)) - \beta(x_3(k)+\gamma)(1-2x_2(k)), \\ x_2(k+1) &= \delta x_2(k)(1-x_2(k)) + \varsigma x_3(k), \\ x_3(k+1) &= \eta((x_3(k)+\gamma)(1-2x_2(k))-1)(1-\theta x_1(k)), \end{aligned} \tag{16}$$

It is known that with some parameters values ($\alpha = 3.8$, $\beta = 0.05$, $\gamma = 0.35$, $\delta = 3.78$, $\varsigma = 0.2$, $\eta = 0.1$, $\theta = 1.9$) the Rössler system (16) exhibits some hyperchaotic dynamic [18].

Adding an input *u* (control law) in (16) consider the next hyperchaotic Rössler system like a *plant P* described by

$$\begin{aligned} x_1(k+1) &= \alpha\, x_1(k)(1-x_1(k)) \\ &\quad - \beta(x_3(k)+\gamma)(1-2x_2(k)) + u(k), \\ x_2(k+1) &= \delta\, x_2(k)(1-x_2(k)) + \varsigma\, x_3(k), \\ x_3(k+1) &= \eta((x_3(k)+\gamma)(1-2x_2(k))-1)(1-\theta\, x_1(k)), \\ y(k) &= x_2(k). \end{aligned} \tag{17}$$

Also considered another hyperchaotic Rössler system like a *model M* described by

$$\begin{aligned} x_{M1}(k+1) &= \alpha\, x_{M1}(k)(1-x_{M1}(k)) \\ &\quad - \beta(x_{M3}(k)+\gamma)(1-2x_{M2}(k)) + u_M(k), \\ x_{M2}(k+1) &= \delta\, x_{M2}(k)(1-x_{M2}(k)) + \varsigma\, x_{M3}(k), \\ x_{M3}(k+1) &= \eta((x_{M3}(k)+\gamma)(1-2x_{M2}(k))-1) \\ &\quad \times (1-\theta\, x_{M1}(k)), \\ y_M(k) &= x_{M2}(k). \end{aligned} \tag{18}$$

we considered the same value parameters in the plant *P* and in the model *M* and in this case $u_M = 0$ to keep the model (18) with hyperchaotic dynamic. The relative degree of the plant (17) and of the model (18) are $d = d_M = 2$, with this we assure that the model-matching problem has solution according with (12).

To solve the model-matching problem following Sec. 3, we defined an auxiliary system according with (6) where the output is the error between the outputs of (18) and (17), then the output of this auxiliary system is given by

$$y_E(k) = y(k) - y_M(k) = x_2(k) - x_{M2}(k).$$

Defining $\zeta_1(k) = y_E(k)$ we show the systems (17) and (18) in a new coordinates taken the form

$$\begin{aligned} \zeta_1(k+1) &= y_E(k+1) = \zeta_2(k), \\ \zeta_2(k+1) &= y_E(k+2) = \zeta_3(k), \\ \zeta_3(k+1) &= y_E(k+3) = v(k), \\ &= -\alpha_2\zeta_3(k) - \alpha_1\zeta_2(k) - \alpha_0\zeta_1(k). \end{aligned} \tag{19}$$

In this case (19) is fully linearizable (if the system is in different form will be necessary to do another stability analysis to assure output synchronization [21]) and then with a proper selection of values α_i $(|\alpha_i| < 1)$ we assure then the closed-loop system will be exponentially stable and condition (3) holds.

With the system (19) we obtained a discrete control law given by (7) that solved model-matching problem. Using this control law (7) we doing the next simulations described here:

We choose $\alpha_i = 0.1$ and with this value the *synchronization time* τ (the time when synchronization was achieved) was in *k*=18. The initial condition of $x(k)$ and $x_M(k)$ were $(-0.3,-0.2,0.3)$ and $(0.1,0.2,-0.1)$ respectively. Figure 1 shows the matching about $y(k)$ and $y_M(k)$, and also shows the synchronization error evolution between outputs of (17) and (18), We can see, after some transient behavior, that synchronization is achieved.

Figure 2 shows the error evolutions in the rest states. Note that the control law *u* only assure the synchronization of $y(k)$ and $y_M(k)$, but in this case we have fully synchronization (synchronization in all states).

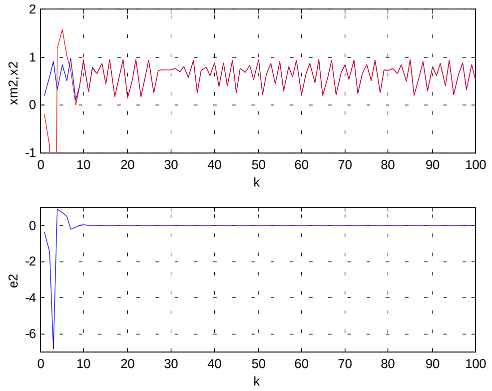

Fig.1. Matching between $y(k)$ and $y_M(k)$, and synchronization error $e_2 = x_2(k) - x_{M2}(k)$.

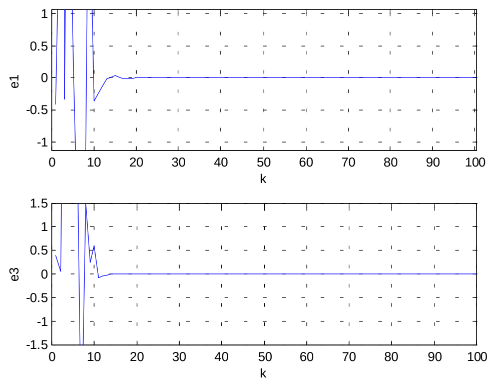

Fig.2. Synchronization error $e_1 = x_1(k) - x_{M1}(k)$ and $e_3 = x_3(k) - x_{M3}(k)$.

In the next simulation we increase the value to $\alpha_i = 0.75$ and we observed that synchronization time τ increase, so we can changes the synchronization time moving the gain values α_i.

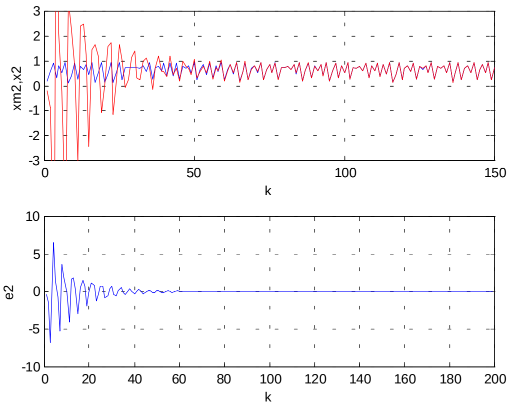

Fig.3. Matching between $y(k)$ and $y_M(k)$, and synchronization error $e_2 = x_2(k) - x_{M2}(k)$.

Figure 3 shows the matching about $y(k)$ and $y_M(k)$, and also shows the synchronization error evolution between outputs of (17) and (18), in this case the synchronization time was in $\tau = 91$.

5 Concluding Remarks

In this paper we have presented an approach to achieve synchronization of hyperchaotic discrete systems via the matching-model approach. In particular this method is indeed suitable to synchronized two hyperchaotic Rössler systems unidirectionally coupled, we show in this case that fully synchronization is achieved.

In a forthcoming article we will be concerned with increase this job using systems not fully linearizable, systems with perturbed signals, and using nolinear control tools to apply this approach in private communication.

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