Regularization of Applied Inverse Problems by the Full Spline Approximation Method

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Abstract: - The operator equation of the first kind is considered as a mathematical model for some applied inverse problems. The input data are given in discrete and noized form, that make the problem of the solution of the operator equation ill-posed. The regularization based on the Full Spline Approximation Method (F.S.A.M.) is proposed. It consists in the recursive using of four steps: 1) pre-smoothing the right-hand side of the equation; 2) application, possibly with precondition, a spline collocation scheme (pre-reconstruction); 3) post-smoothing of the pre-reconstructed solution; 4) checking up the stop rule. The F.S.A.M. differs from the previously proposed and justified by the author Spline Approximation Method (S.A.M.) by: 1) the presence of the precondition and the post-smoothing; 2) realisation of the pre- and post-smoothing in the spline spaces of possiblly different dimensions, that leads proposed F.S.A.M. in a class of multigreed methods. The new element in the proposed method is considering the number of the recursions and the precondition parameter as two independent regularization parameters. The theoretical foundation of F.S.A.M. so as the results of numerical experiments for some integral equation of electrodynamics and inverse problem of electroencephalography are presented.

Key-words: - regularization, operator and integral equations of the first kind, spline, collocation, smoothing.

1. Introduction

Let a linear compact injective [1] operator A, acting from the Gilbert space G into G, is given. Let $l_i, i = 1, ..., n$, be limited linear independent functionals and $\mathcal{L}_n$ is the operator from G into R_n : $\mathcal{L}_n g \equiv (l_1(g), ..., l_n(g))^T$. A problem of the approximate solution of an operator equation

$$Au = f \quad (1)$$

on the known values $l_i(f)$, $i = 1, ..., n$; is considered. Let the right-hand side f be representative in a kind of a sum of an exact right-hand side $\overline{f}$ and an error $\zeta : f = \overline{f} + \zeta$, with a given estimation $\|\mathcal{L}_n \zeta\|_{R_n} / \sqrt{n} \leq \delta$. We will propose that $\overline{f} \in R(A)$ - the range of the operator A, $\zeta \notin R(A)$, thus, the problem (1) is incorrect (unstable) [2]. Let us choose the Hilbert space $U \subseteq G$ and a linear closed operator $L : U \to G$. The element v_n is named by interpolating spline [2] for given $v \in G$, if it is a solution of a problem: $\|Lv_n\|_G = \inf_{g \in U(v)} \|Lg\|_G$, $U(v)$

$= \{g \in U : \mathcal{L}_n g = \mathcal{L}_n v\}$. We will assume that conditions of existence, uniqueness of the interpolating spline are executed [2]. Hence, we can determine an operator $\mathcal{P}_n$ of spline - interpolation, that the vector $\mathcal{L}_n v$ for any element v associates with a spline v_n. It is known [2], that the set of all splines forms the n - dimensional Hilbert subspace $U_n \subset U$ with some basis $\varphi_1, ..., \varphi_n$. The space U_n is complete concerning the norm $\|u\|_{L,n} \equiv (u,u)_{L,n}^{1/2}$, where $(u,v)_{L,\,n} \equiv \sum_{i=1}^{n} l_i(u) l_i(v) + (u,v)_L$, $(u,v)_L = (Lu, Lv)_G$, $u, v \in U$. The spline v_n, interpolating the element v, has an orthogonal property [2]: $(r_n, v_n - v)_{L,n} = 0$ for any $r_n \in U_n$. We will assume that "complementary" $\gamma \|g\|_G \leq \|g\|_{L,n}$, condition $g \in U$, is fulfilled for some $\gamma = const > 0$, and the convergency of the interpolating spline takes place, so that $\varrho_n \equiv ||\mathcal{P}_n \mathcal{L}_n g - g||_{L,\,n} \to 0$, $n \to \infty$, $g \in U$ [2]. We shall consider in general case bellow the different spline-subspaces $U_{n_l}^{(l)}$ with bases $\{\varphi_j^{(l)}\}$ of different dimensions n_l for the collocation $(l = 0)$, for the pre-smoothing $(l = 1)$ and for the post-smoothing $(l = 2)$, corresponding to constructive operators $L_{n_l}^{(l)}$, $\mathcal{L}_{n_l}^{(l)}(g) = \left(l_1^{(l)}(g), ..., l_{n_l}^{(l)}(g)\right)^T$, spline-interpolation operators $\mathcal{P}_{n_l}^{(l)}$. Realization of smoothings is based on the explicit [4] approximation operators $P_{n_l}^{(l)}$ acting from R_{n_l} into $U_{n_l}^{(l)}$: $P_{n_l}^{(l)} \mathcal{L}_{n_l}^{(l)} f = \sum_{j=1}^{n_l} l_j^{(l)}(f) \varphi_j^{(l)}$, so as $\left\| P_{n_l}^{(l)} \mathcal{L}_{n_l}^{(l)} g - g \right\|_G \to 0$, $n = \min_{0 \leq\, l\, \leq\, 2} n_l \to \infty$. We assume also that the matrix $V_{n_l}^{(l)} = \mathcal{L}_{n_l}^{(l)} P_{n_l}^{(l)}$ has the eigenvalue spectrum $\lambda_i : 0 < \lambda_i \leq 1$, $i = 1, ..., n_l$. To construct the collocation scheme we need approximate the operator A by some operator A_{n_0}, acting from G into G, and also to use some approximation of the right-hand side. For example, if A is an integral operator, A_{n_0} means an operator with approximate kernel. The collocation means, that the approximate solution of the equation (1) with the approximated operator and the right-hand side is located in a kind $u_{n_0} = \sum_{i=1}^{n_0} c_j \varphi_j^{(0)}$, where c_i are unknown coefficients, determining from the preconditioned collocation scheme. Moreover, we will use iterative scheme and assume that numbers $n_l = n_l(k)$ are dependent from the number of iteration k, that leads proposed F.S.A.M. in a class of multigrid methods. The foundation of the F.S.A.M. consist in: 1) justification of existence and uniqueness of the preconditioned collocation spline; 2) justification of the regularization properties for appropriated choice of the regularization parameters. In a paper [5] we generalized well known [1] and previous results of the author [3], [4] on foundations of the collocation scheme for the case without the proposition about transferring by the operator A the linear span of the basic functions into itself and also, when the operator A_{n_0} approximates the operator A in a hole in the space G without eliminating it's calculating exactly main part. We justify here the existence and the uniqueness of the collocation spline with the precondition and generalize the regularization properties for the case of the presence of pre- and post-smoothing and combination with the precondition by the Lavrentiev regularization (L.R.).

2. Formulation of the Full Spline Approximation Method and it's Foundation

The *Full Spline Approximation Method* for the numerical solution of the problem (1) consists in constructing a recuperation spline $u_{n,\delta}$ by recursive using of four steps:

1) *pre-smoothing* the right-hand side $f \in F$ of the equation (1), i.e., calculation of an element y_k: $y_k = P_{n_1(k)}^{(1)} \mathcal{L}_{n_1(k)}^{(1)}\ y_{k-1}$; $y_0 = f$;

2) *pre-reconstruction* by a preconditioned collocation method for the equation (1) with the approximated operator and smoothed right-hand side, i.e., solving the preconditioned by the L.R. system

of linear algebraic equations $(\mathcal{A}_{n_0}+\alpha_k I)\hat{c}^{(k)}=\hat{f}_k$ with a matrix $\mathcal{A}_{n_0}=(\widetilde{a}_{ij})$, $\widetilde{a}_{ij}=l_i^{(0)}(A_{n_0}\varphi_j^{(0)})$, $i,\ j=1,...,n$; I is the identical matrix, α_k is a positive numerical parameter, the right-hand side $\hat{f}_k=\mathcal{L}_{n_0}^{(0)}(y_k)$, relatively components of a vector $\hat{c}^{(k)}=(c_1^{(k)},...,c_{n_0}^{(k)})^T$ used in the location $u_{n_0(k)}=\sum_{i=1}^{n_0(k)}c_j^{(k)}\varphi_j^{(0)}$ of the collocation spline;

3) *post-smoothing*, i.e. calculation the splines $\widetilde{u}_{n_2(k)}=P_{n_2(k)}^{(2)}\mathcal{L}_{n_2(k)}^{(2)}u_{n_0(k)}$, $y_k=P_{n_0(k)}^{(0)}(\mathcal{A}_{n_0}+\alpha_k I)\mathcal{L}_{n_0(k)}^{(0)}\widetilde{u}_{n_2(k)}$;

4) *stop rule comparing:* if the discrepancy $r_k\equiv||\mathcal{L}_{n_1(1)}^{(1)}f-\mathcal{L}_{n_1(1)}^{(1)}\ y_k||\ /\sqrt{n_1(1)}\le\delta$, then $k:=k+1$, go to the item 1); if $r_k>\delta$, then $K=k$, $u_{n,\delta}=\widetilde{u}_{n_2(K)}$, stop.

We propose that the exact solution of the equation (1) with the exact right-hand side $\overline{u}=A^{-1}\overline{f}\in U^{(0)}$.

Theorem. Let the hypothesis are fulfilled:

1) $A=A^*>0$ in G;

2) $\left(L^{(0)}Ag;L^{(0)}g\right)_G=\left(AL^{(0)}g;L^{(0)}g\right)_G$, $\left(L^{(0)}A_{n_0}g;L^{(0)}g\right)_G=\left(A_{n_0}L^{(0)}g;L^{(0)}g\right)_G$, for any $g\in U^{(0)}$;

3) $||A-A_{n_0}||_{G\to G}=\gamma_{n_0}\to 0$, $n_0\to\infty$;

4) $\gamma_A\,\|g\|_G^2\le(Ag,g)_{L^{(0)},n_0}$, for any $g\in U^{(0)}$, $\gamma_A=const>0$;

5) $\alpha_k=\alpha=\max\{\sqrt{\gamma_{n_0}},\ \sqrt{\delta}\}$;

then for sufficiently large n and small δ the recuperation spline $u_{n,\delta}$ exists, is unique and has the regularization property $\lim_{n\to\infty,\delta\to 0}||\overline{u}-u_{n,\delta}||_G=0$.

Proof. The justification of the existence and uniqueness of the collocation spline without the precondition under the hypothesis 1) - 3) is presented in [5]. It means that the matrix $\mathcal{A}_n$ is invertible, then for sufficiently small $\alpha_k>0$ the matrix $(\mathcal{A}_n+\alpha_k I)$ is invertible too, hence the recuperation spline $u_{n,\delta}$ exists and is unique. For justification of the regularization property we note the first, that the operators $P_{n_l(k)}^{(l)},l=1,2$ realize the smoothing and eliminate errors from the input data and from the pre-reconstructed spline. For example, in the case $n_l(k)=n(k),l=0,1,2$; $L^{(1)}=L^{(2)}$, the matrixes $\mathcal{A}_n$, $V_n^{(1)}=V_n^{(2)}=V$ are symmetrical, the procedures 1) - 3) of the F.S.A.M. can be presented in the following matrix form: $(\mathcal{A}_{n_0}+\alpha_k I)V_n^{(2)}(\mathcal{A}_{n_0}+\alpha_k I)^{-1}V_n^{(1)}=(V)^2$. Hence, the foundation of the regularization properties of the pre- and post-smoothings can be obtained from the results, presented in [3], [4]. But here the precondition by the L.R. play the main part in the regularization process for the space G. The smoothings are important for recuperation in spaces with more strong norms, or, by the other words, for improvement of the recuperation, as we will demonstrate in numerical experiments bellow. Let us designate $u_{n_0}^0$ the collocation spline, constructed without the precondition on the exact operator A and the exact right-hand side of the equation (1). From the results obtained in [3] it follows the existence, uniqueness of $u_{n_0}^0$ and it's convergency to $\overline{u}$ in G, when $n_0\to\infty$. The vectors $\widehat{c}_0$, $\widehat{c}_{n,\delta}$ of coefficients of the splines $u_{n_0}^0$ and $u_{n,\delta}$ in their expansions with the basis $\{\varphi_i^{(0)}\}$ satisfy correspondingly to the systems of linear algebraic equations $\mathcal{A}\widehat{c}_0=\mathcal{L}_{n_0}^{(0)}\overline{f}$, $(\mathcal{A}_{n_0}+\alpha_K I)\widehat{c}_{n,\delta}=\hat{f}_K$, $\mathcal{A}=\{a_{ij}^0\}$, $a_{ij}^0=l_i(A\varphi_j^{(0)})$, $i,\ j=1,...,n_0$. Due to the discrepancy principle we have the estimation $||\mathcal{L}_{n_0}^{(0)}\overline{f}-\hat{f}_K||_{R_{n_0}}/\sqrt{n_0}\le 2\delta$. The matrixes of these systems satisfy to estimation $||\mathcal{A}-\mathcal{A}_{n_0}||\le\gamma_{n_0}$. From the well-known properties of the L.R. the relations follow: $||\widehat{c}_0-\widehat{c}_{n,\delta}||_{R_{n_0}}/\sqrt{n_0}\le O(\frac{\delta+\gamma_{n_0}}{\alpha})\le O(\sqrt{\delta}+\sqrt{\gamma_{n_0}})\to 0$, $n\to\infty$, $\delta\to 0$. This implies the stability in the space G and complete the proof.

3. Solving the Integral Equations with the Singularity in the Kernel

Let us consider a singular integral equation of the first kind:

$$Au \equiv \int_0^{2\pi} K(x,t)u(t)dt = f(x) \qquad (2)$$

for the periodic on the segment $[0, 2\pi]$ functions u and f. The kernel $K(x,t) = -log(\sin^2(0.5(x-t)))$ has the logarithmic singularity. These integral equation is the characteristic one for a number of applied inverse problems in electrodynamics [1], [4].

As the subspaces of spline functions $U_{n_0}^{(0)}$ we use bellow the linear periodic on the segment $[0, 2\pi]$ splines and as $U_{n_l}^{(l)}, l = 1, 2$ - the cubic periodic splines. We introduce uniform grids $t_i = (i-1)h_l$, $i = -2, ..., n_l + 2$ with a steps $h_l = 2\pi/(n_l - 1)$, $l = 0, 1, 2$; functionals $l_i(u) = f(t_i)$, $i = 1, ..., n_l$; $G = L_2[0, 2\pi]$, operator $L^{(0)}f(t) = df(t)/dt$, $U^{(0)} = W_2^{(1)}[0, 2\pi]$; $L^{(l)}f(t) = d^2f(t)/dt^2$, $U^{(l)} = W_2^{(2)}[0, 2\pi]$, $l = 1, 2$. As a basis we use the Schoenberg B-splines of the $m-$th degree $s_{j,m}(t)$, $m = 0, 1, 3$. So we use in the collocation scheme the linear splines $u_1(t) = \sum_{j=1}^{n_0} c_j s_{j,1}(t)$, $t \in [0, 2\pi]$ with unknown coefficients that obey the condition: $c_1 = c_{n_0}$. Let us introduce auxiliary uniform grid $\{t_{i+1/2}\} : t_{i+1/2} = t_i + (t_{i+1} - t_i)/2$. We will use approximations A_{n_0} of the operator A constructed by the following formulas: $A_{n_0}u = \sum_{i=1}^{n_0-1} l_i^{(h)}(\bar{A}\ u)s_{i,0}(x)$, where $\bar{A}\ u = h_0 \sum_{i=1}^{n_0-1} K(x, t_i)u(t_i)$, $l_i^{(h)}u = u(t_{i+1/2})$. It is easy to check up that all hypothesis of the Theorem above are fulfilled and proposed F.S.A.M. give the regularization algorithm in $L_2[0, 2\pi]$. We have realized this algorithm as a collection of MATLAB programs. Let us put outcomes of some numerical experiments for the exact right-hand side $\overline{f}(x) = \pi \sin x$; exact solution $\overline{u}(t) = \sin t$, the noized right-hand side $f(x_j) = \overline{f}(x_j) + \xi_j$, $j = 1, ..., n_1$, $\{\xi_j\}$ are causal errors, estimated by δ. We fixed $n_l(k) = n$ independent from the number of iterations k and calculated for different $\delta = \varepsilon_i = 0.02i$, $i = 1, ..., 15$ the discrete quadratic mean errors $r(\varepsilon_i)$ of the recuperation of the integral equation solution. The numerical results for different n demonstrate, that, in general, the recuperation by L.R. with pre-smoothing (S.A.M.) is better than the recuperation by L.R. only, but the recuperation by L.R. with pre- and post-smoothing (F.S.A.M.) is the best. The graphics of the corresponding $r(\varepsilon)$ for $n = 31$ are presented in Fig. 1. We note that the regularization properties of the F.S.A.M. are justified in Theorem above for arbitrary dependence $n_l(k)$ from k, such that $n_l(k) \geq n_l(1)$, that are sufficiently large. Some numerical experiments demonstrate the improvement of the recuperation, if $n_l(k)$ increase with k. The choice of $n_l(k)$, that guarantees this improvement is under the author's investigation.

4. EEG Inverse Problems

Very important area of the constructed algorithms application is Electroencephalography (EEG) Inverse Problems. We use an approach based on the physical model presenting a spatial distribution of the potential fields caused by the neuronal current J on the cerebral cortex only. Let us consider a model of a head occupying an area Ω, consisting from two subareas: $\Omega = \bigcup_{i=1}^{2} \Omega_i$, that are restricted by two concentric spheres S_1 and S_2 with radiuses r_1 and r_2. Subarea Ω_1 correspond to the brain, and Ω_2 - to the rest part of a head, with conductivities σ_1 and σ_2. We will design as $\vec{n}_j$ vectors of the exterior unit normal for the area Ω_j on S_1, $v = v_j$ - potentials of the electric field in Ω_j, $j = 1, 2$. Under made assumptions it is possible to write the EEG inverse problem as a problem of calculating (recuperation)

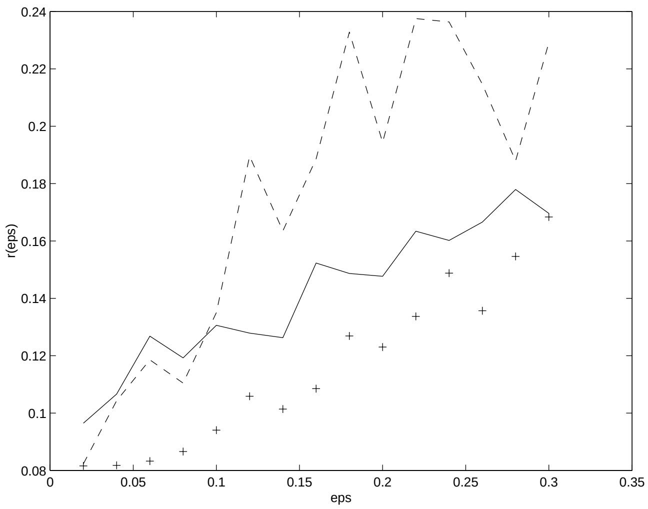

Figure 1: Errors of the recuperation of the one-dimensional integral equation solution: "- -" - by L.R.; "-" - by S.A.M.; "+" - by F.S.A.M.

the normal current

$$J = \sigma_1 \frac{\partial v_1}{\partial \vec{n}_1} - \sigma_2 \frac{\partial v_2}{\partial \vec{n}_2}, \tag{3}$$

where v_i are solutions of a boundary problem:

$$\Delta v_i = 0, \quad \text{in } \Omega_i, \; i = 1, 2; \tag{4}$$

$$v_1 = v_2 \quad \text{in } S_1, \tag{5}$$

$$v_2 = f \quad \text{in } S_2, \tag{6}$$

where f is the potential of the electrical field measured on the scalp. Using a potential theory we can present $v(P), P \in \Omega$ in a form of the potential of the simple ley with unknown density $u(M)$, $M \in S_1$, and obtain respectively $u(M)$ an integral equation

$$Au \equiv \frac{1}{4\pi} \int\int_{S_1} \frac{u(M)}{r(P,M)} dS_1(M) = f(P), \tag{7}$$

where $P \in S_2$, $r(P,M)$ is a distance between points M and P. We will locate in a spherical system of coordinates the density $u(\theta, \varphi)$ as a bilinear periodic spline $s(\theta,\varphi) = \sum_{i=1}^{N} \sum_{j=1}^{M} c_{i,j} \; s_{i,1}(\theta) \; s_{j,\,1}(\varphi)$, $[\theta, \varphi] \in D \equiv [0, 2\pi] \times [0, \pi]$. B-splines are constructed on the corresponding uniform grids. The pre- and post-smoothing are realized by bicubic periodic splines. The approximation of the operator A is constructed similarly to the approximation of the integral operator of the equation (2). It is possible to choose the corresponding spaces and check up the fulfillment of all hypothesis of the Theorem, that guarantees the regularization of the density recuperation by the F.S.A.M. in the space $L_2[D]$, hence, the stability of the current recuperation.

Let us put outcomes of some numerical experiments on the model example, that confirm the stability of the current recuperation by proposed algorithm. We supposed $r_1 = 3$, $r_2 = 4$; $\sigma_1 = 5$, $\sigma_2 = 1$ and solved the first the direct problem, when for given $J = \sin(\theta + \varphi)$ we calculated approximately coefficients of the spline $s(\theta, \varphi)$, considered as solution of the corresponding to the formula (3) the Fredholm integral equation of the second kind. Obtained coefficients we used to calculate the "exact" values of the potential on the sphere S_2. Then we formed noised data of the right-hand side of the equation (7) by adding causal errors to calculated "exact" values. We used the simple variant of F.S.A.M. with the same grids in recursions with number of units $N = 21$, $M = 21$. We demonstrate in a Fig. 2 the recuperation of the current J on the sphere S_1 for $\delta = 0.3$. We have the best results of reconstruction by F.S.A.M.

Some results of S.A.M. application to the coefficient inverse problems of heat-conduction, identifying of characteristics of the porous media of confined aquifers, and some another EEG inverse problems are presented in [5], [6], [7].

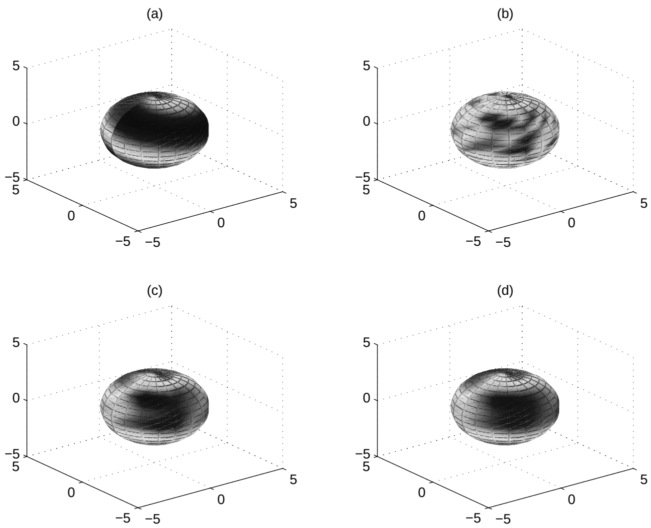

Figure 2: The recuperation of the current J in EEG inverse problem: graph (a) – the exact J; graph (b) – recuperation by L.R. only; graph (c) – recuperation by S.A.M.; graph (d) – recuperation by F.S.A.M.

References:

[1] R. Kress. *Linear Integral Equations.* Springer-Verlag, Berlin, 1989.

[2] V. A. Morozov. *Regularization Methods for Ill-Posed Problems.* CRC Press, London, 1993.

[3] A. I. Grebennikov. Solving Integral Equations with a Singularity in the Kernel by Spline Approximation Method. *Sov. J. Numer. Anal. Mathem. Modelling.* Vol. **5**, N 3, p. 199-208 (1990).

[4] V. A. Morozov and A. I. Grebennikov. *Methods of Solving Ill-Posed Problems: Algorithmic Aspect.* Moscow, Moscow State Univ. Publ. House, 1992.

[5] A. I. Grebennikov. Solución de Problemas Mal - Planteados por Método de Aproximación Spline y Aplicación. *Métodos Numéricos en Ingeneria y Ciencias Aplicadas.* CIMNE, Barselona, 2001.

[6] A. I. Grebennikov. Método de Aproximación Spline para Resolver Problemas Inestables y su Aplicación. *Rev. Aportaciones Matematicas, Serie Comunicaciones.* 27, p. 53-70, 2000.

[7] A. Grebennikov. Spline Approximation Method of Solving Some Coefficient Inverse Problems for Differential Equations of the Parabolic Type. *Inverse Problems in Engineering J.*(2001), (to appear)