A Discrete Fractional Gabor Expansion for Time–Frequency Signal Analysis

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Abstract: - In this work, we present a discrete fractional Gabor representation on a general, non-rectangular time-frequency lattice. The traditional Gabor expansion represents a signal in terms of time and frequency shifted basis functions, called Gabor logons. This constant-bandwidth analysis uses a fixed, and rectangular time-frequency plane tiling. Many of the practical signals require a more flexible, non-rectangular time-frequency lattice for a compact representation. The proposed fractional Gabor method uses a set of basis functions that are related to the fractional Fourier basis and generate a non-rectangular tiling. Simulation results are presented to illustrate the performance of our method.

Key-Words: - Time-frequency analysis, Gabor expansion, Fractional Fourier Transform.

1 Introduction

Time–frequency (TF) analysis provides a characterization of signals in terms of joint time and frequency content [1]. One of the fundamental issues in the TF analysis is obtaining the distribution of signal energy over joint TF plane with a delta function concentration [1]. The discrete Gabor expansion is a TF signal decomposition which represents a signal in terms of time and frequency translated basis functions called TF atoms [2, 3]. Gabor basis functions $g_{m,k}(n)$ are obtained by shifting and modulating with a sinusoid a single window function $g(n)$, which results in a fixed and rectangular TF plane tiling. However, if the signal to be represented is not modeled well by this constant-bandwidth analysis, its Gabor representation displays poor TF localization [4, 5, 6]. Many of the practical signals such as speech, music, biological, and seismic signals have time-varying frequency nature that is not appropriate for sinusoidal analysis [4, 6]. Thus the traditional Gabor expansion of such signals will require large number of coefficients yielding a poor TF localization. The compactness of the Gabor representation is improved if the basis functions match the time-varying frequency behavior of the signal [6, 7, 8]. Here we present a new, fractional Gabor expansion that uses a more flexible, non–rectangular TF lattice. The basis functions of the proposed expansion are related to the fractional Fourier basis.

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2 The Discrete Gabor Expansion

The traditional Gabor expansion [2, 3] represents a signal in terms of time and frequency shifted basis functions, and has been used in various applications to analyze the time–varying frequency content of a signal [9]. Basis functions of the Gabor representation are obtained by translating and modulating with sinusoids a single window function.The discrete Gabor expansion of a finite-support signal $x(n),\ 0 \leq n \leq N-1$ is given by [3]

$$x(n) = \sum_{m=0}^{M-1} \sum_{k=0}^{K-1} c_{m,k}\, \tilde{g}_{m,k}(n) \tag{1}$$

where the basis function

$$\tilde{g}_{m,k}(n) = \tilde{g}(n - mL)\, e^{j\omega_k n} \tag{2}$$

and $\omega_k = 2\pi k L'/N$. The Gabor expansion parameters M, K, L, and L' are positive integers constrained by $ML = KL' = N$ where M and K are the number of analysis samples in time and frequency, respectively, and L and L' are the time and frequency steps, respectively. Existence, uniqueness and numerical stability of the representation depend on the choice of parameters L and L'. For numerically stable representations, L and L' must satisfy $L\,L' \leq N$, or equivalently that $L \leq K$. The case where $L = K$, is called the critical sampling, and the case $L < K$ is called the oversampling. The synthesis window $\tilde{g}_(n)$ is a periodic extension (by N) of $g_(n)$ which is normalized to unit energy for definiteness [3].

In general, the set of time and frequency shifted window functions, i.e., Gabor logons, $\{\tilde{g}_{m,k}(n)\}$ forms a non–orthogonal basis for the square–summable sequences space $\ell^2(\mathcal{R})$. Hence the calculation of the Gabor coefficients is not a simple task since projection by the usual inner product cannot be used. One of the methods [3], uses an auxiliary function $\gamma(n)$ called the biorthogonal window or dual function of $g(n)$. Then the Gabor coefficients $\{c_{m,k}\}$ can be evaluated by

$$c_{m,k} = \sum_{n=0}^{N-1} x(n)\, \tilde{\gamma}^*_{m,k}(n) \tag{3}$$

where the analysis functions are

$$\tilde{\gamma}_{m,k}(n) = \tilde{\gamma}(n - mL)\, e^{j\omega_k n} \tag{4}$$

where again $\tilde{\gamma}(n)$ is a periodic version of the dual window $\gamma(n)$. Completeness condition of the basis set is obtained by substituting (3) into (1) to get that

$$\sum_{m=0}^{M-1} \sum_{k=0}^{K-1} \tilde{g}_{m,k}(n)\, \tilde{\gamma}^*_{m,k}(\ell) = \delta(n - \ell) \tag{5}$$

where $\delta(\cdot)$ denotes the Dirac delta function. The above completeness relation yields equivalent but simpler biorthogonality condition between the analysis and synthesis basis sets via the discrete Poisson-sum formula [3]:

$$\sum_{n=0}^{N-1} \tilde{g}(n + mK) e^{-j\frac{2\pi}{L}kn}\, \tilde{\gamma}^*(n) = \frac{L}{K}\, \delta_m \delta_k \tag{6}$$

for $0 \leq m \leq L' - 1, \quad 0 \leq k \leq L - 1$. The analysis window $\gamma(n)$ is obtained by solving the equation system of the above biorthogonality condition.

Gabor analysis basis $\{\tilde{\gamma}_{m,k}(n)\}$ with a fixed window and sinusoidal modulation tiles the time–frequency plane in a rectangular fashion causing a constant bandwidth analysis. Constant bandwidth methods, such as spectrogram [1] and the Gabor expansion provide signal representations with poor time–frequency resolution [4]. Recently, representations on a non–rectangular TF grid has attracted a considerable attention [6, 10]. A non–rectangular lattice is more appropriate for the TF analysis of signals with time–varying frequency content. Thus the motivation for a fractional Gabor analysis.

3 A Fractional Gabor Expansion

We define a discrete fractional Gabor expansion for $x(n),\ 0 \leq n \leq N - 1$, as follows:

$$x(n) = \sum_{m=0}^{M-1} \sum_{k=0}^{K-1} c_{m,k,\alpha}\, \tilde{g}_{m,k,\alpha}(n) \tag{7}$$

where $c_{m,k,\alpha}$ are the fractional Gabor coefficients, α is the order of the fraction, and the basis functions are

$$\tilde{g}_{m,k,\alpha}(n) = \tilde{g}(n - mL)\, W_{\alpha,k}(n)$$

Here $\tilde{g}(n)$ is a periodic version of a unit energy Gabor window $g(n)$ and $W_{\alpha,k}(n)$ is the fractional kernel,

$$W_{\alpha,k}(n) = e^{j[-\frac{1}{2}(n^2 + (\omega_k \sin\alpha)^2)\cot\alpha + \omega_k n]}$$

where $\omega_k = 2\pi k/K$. The kernel above is similar to the Fractional Fourier Series basis functions [11]. The expansion in (7) reduces to the traditional Gabor for $\alpha = \pi/2$. The parameters M, K, L, and L', are same as in the traditional Gabor expansion. In our derivations, we always consider the general, oversampled case, i.e., $L < K$. The Gabor coefficients can be evaluated as before by

$$c_{m,k,\alpha} = \sum_{n=0}^{N-1} x(n)\, \tilde{\gamma}^*_{m,k,\alpha}(n) \tag{8}$$

where the analysis functions are

$$\tilde{\gamma}_{m,k,\alpha}(n) = \tilde{\gamma}(n - mL)\, W_{\alpha,k}(n)$$

and $\tilde{\gamma}(n)$ is periodic version of a $\gamma(n)$ that is solved from a fractional biorthogonality condition between $g(n)$ and $\gamma(n)$.

The completeness condition for the fractional Gabor basis, is obtained by substituting (8) in (7),

$$\begin{aligned} x(n) &= \sum_{m=0}^{M-1} \sum_{k=0}^{K-1} \left(\sum_{\ell=0}^{N-1} x(\ell)\, \tilde{\gamma}^*(\ell - mL)\, W^*_{\alpha,k}(\ell) \right) \\ &\times \quad \tilde{g}(n - mL) W_{\alpha,k}(n) \\ &= \sum_{\ell=0}^{N-1} \sum_{m=0}^{M-1} \sum_{k=0}^{K-1} \tilde{g}(n - mL) \tilde{\gamma}^*(\ell - mL) \\ &\times \quad e^{j[-\frac{1}{2}(n^2 - \ell^2)\cot\alpha + \omega_k (n-\ell)]} \end{aligned}$$

Then we obtain that the windows must satisfy the following completeness relation:

$$\sum_{m=0}^{M-1} \sum_{k=0}^{K-1} \tilde{g}(n - mL) \tilde{\gamma}^*(\ell - mL) e^{j[-\frac{1}{2}(n^2 - l^2)\cot\alpha]} \times e^{j\,\omega_k (n-\ell)} = \delta(n - \ell) \qquad (9)$$

The fractional biorthogonality condition that we need to solve the analysis or dual function $\gamma(n)$ is obtained from the above completeness relation using discrete Poisson sum formula as

$$\begin{aligned} &\sum_{n=0}^{N-1} \tilde{g}^*(n + mK) e^{jk\frac{2\pi}{L}(n+mK)} \tilde{\gamma}(n) \\ &\quad \times e^{j(nmK + \frac{m^2K^2}{2})\cot\alpha} = \frac{L}{K}\, \delta_m \delta_k \\ &0 \le m \le L' - 1, \quad 0 \le k \le L - 1 \qquad (10) \end{aligned}$$

Completeness and biorthogonality conditions given in equations (9) and (10) reduce to the conditions in the traditional case [3] for $\alpha = \pi/2$. This indicates that the above fractional expansion is a generalization of the discrete Gabor expansion. In Fig. 1, we show a Gauss window $g(n), n = 0, 1, \cdots, 127$ on the top figure, and its biorthogonal $\gamma(n)$ for two different set of sampling parameters obtained by solving equation (10) with $\alpha = \pi/4$. The window in the middle is obtained using $L = 16, K = 16$ that is the critical sampling and the window at the bottom is calculated with $L = 8, K = 64$ as an example of the oversampling.

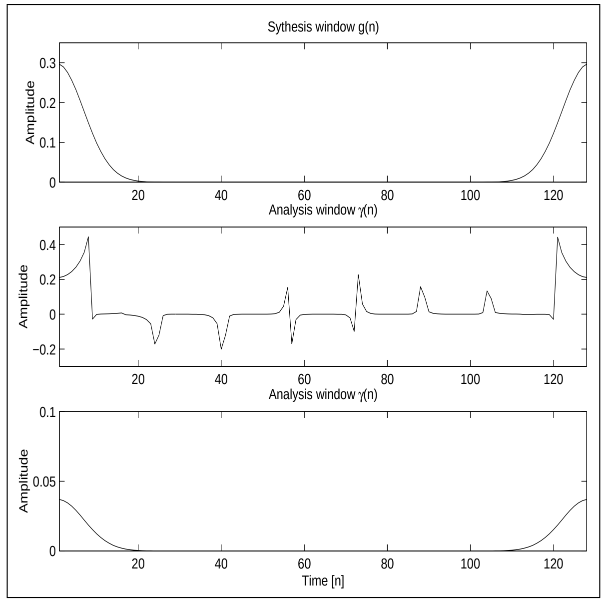

Fig. 1. A Gauss synthesis window (top figure), and its biorthogonal windows in critical (middle) and oversampling (bottom) cases.

4 Simulation Results

We consider a signal composed of two linear chirps. Using our fractional Gabor method, we analyzed the signal with two different fractional orders. Figs. 2 and 3 show the magnitude squared fractional Gabor coefficients, $|c_{m,k,\alpha}|^2$, of this two-chirp signal with $\alpha = \pi/4$ and $\alpha = 3\pi/8$ respectively. Notice that, the component that is matched by the analysis angle becomes a narrow-band signal and represented with higher resolution.

5 Conclusions

In this paper, we present a discrete fractional Gabor expansion on a flexible, non–rectangular TF plane for the analysis of non–stationary signals. We give the completeness and biorthogonality conditions of this new expansion. Simulations show that the fractional expansion gives high resolution representations.

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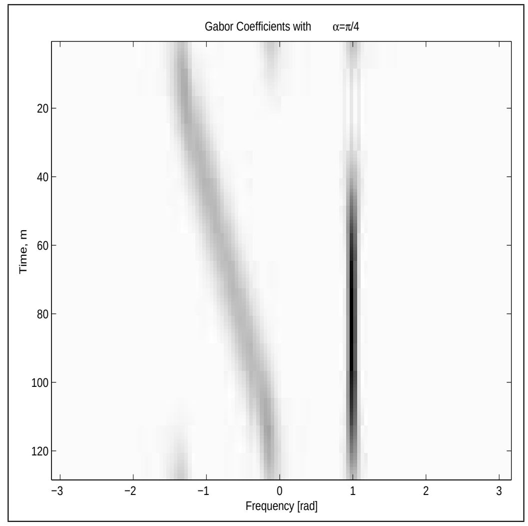

Fig. 2. Gabor coefficients of the two-chirp signal using fractional order $\alpha = \pi/4$.

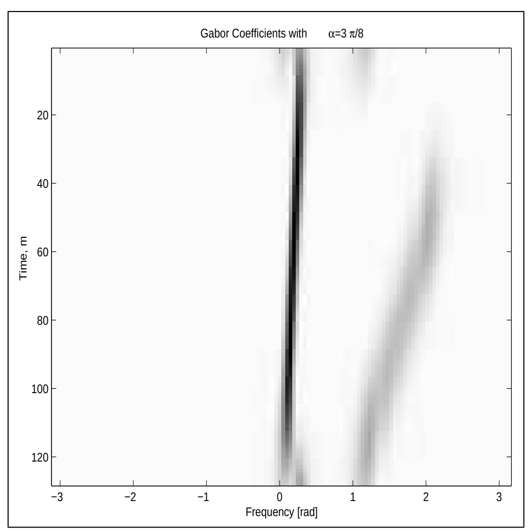

Fig. 3. Gabor coefficients using $\alpha = 3\pi/8$.