The application of isoperimetric inequalities for nonlinear eigenvalue problems

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Abstract: - Our aim is to show the interplay between geometry analysis and applications of the theory of isoperimetric inequalities for some nonlinear problems. Reviewing the isoperimetric inequalities valid on Minkowskian plane we show that we can get estimations of physical quantities, namely, estimation on the first eigenvalue of nonlinear eigenvalue problems, on the basis of easily accessible geometrical data.

Key-Words: - nonlinear eigenvalue problems, isoperimetric inequalities, Minkowskian geometry

1 The classical isoperimetric inequality

The classical isoperimetric inequality after which all such inequalities are named states that of all plane curves of given perimeter the circle encloses the largest area. This extremal property is expressed in the inequality:

$$L^2 \geq 4\pi A, \tag{1}$$

where A denotes the area of the domain and L the length of its boundary curve, and where equality holds only for circles. This inequality was known already to the Greeks. Pappus, in whose writings these results are preserved, attributes their discovery to Zenodorus.

In their famous book *Isoperimetric Inequalities in Mathematical Physics,* Pólya and Szegő extended this notion to include inequalities for domain functionals, provided that the equality sign is attained for some domain or in the limit as the domain degenerates [15].

2 Isoperimetric inequalities in "broader sense"

There are several interesting and important geometrical and physical quantities depending on the shape and size of a curve:

-the length of its perimeter, the area included,

-the moment of inertia, with respect to the centroid, of a homogeneous plate bounded by the curve,

-the torsional rigidity of an elastic beam the cross section of which is bounded by the given curve,

-the principal frequency of a membrane of which the given curve is the rim,

-the electrostatic capacity of a plate of the same shape and size,

-and several other quantities.

By the help of the isoperimetric inequalities we estimate physical quantities on the basis of easily accessible geometrical data.

The study of isoperimetric inequalities in a broader sense began with the conjecture of St Venant in 1856, that of all cylindrical beams of given cross-sectional area the circular beam has the highest torsional rigidity. In 1877 Lord Rayleigh conjectured that of all vibrating elastic membranes of constant density and fixed area the circular membrane has the minimum principal frequency. He gave some evidence to support the conjecture. In 1903 H. Poincaré made the conjecture that of all solids of given volume the sphere has the minimum exterior electrostatic capacity.

The proofs of these conjectures were given later. Around 1923 G. Faber [7] and E. Krahn [10] obtained independently the statement of Rayleigh. The proof were based on the introduction of a special system of curvilinear coordinates. G. Szegő and G. Pólya gave another proof by using the Steiner symmetrization [15]. In recent literature the statement of the Rayleigh conjecture is usually referred to as the Faber-Krahn inequality. This property is expressed by the inequality

$$\lambda_1 \geq \frac{\pi\, j_0{}^2}{A} \tag{2}$$

with equality only for the circle, and where j_0 is the first positive zero of the Bessel function of the first kind $J_0(x)$, moreover A is the area of the domain Ω.

In 1930 G. Szegő gave a rigorous proof of H. Poincaré's conjecture [16].

In 1948 G. Pólya verified the conjecture of B. de St. Venant [14].

3 Geometrical inequalities

The theory of isoperimetric inequalities is a subject of great diversity and complexity. Our aim is to show the interplay between geometry analysis and applications for some nonlinear problems.

3.1 The Bonnesen inequality

Let curve c_ρ be given as follows

$$|x|^{\frac{1}{p}+1} + |y|^{\frac{1}{p}+1} = |\rho|^{\frac{1}{p}+1} \qquad (3)$$

$$\rho \in \mathbf{R}^+, \quad 0 < p < \infty .$$

This is a central symmetric convex curve, which plays the same role as the circle. If $p = 1$, then curve c_ρ is a circle with radius ρ.

The Minkowskian length of curve c_ρ defined by (3) is

$$L_p(c_\rho) = 4 \int_{x=0}^{\rho} \sqrt[p+1]{1 + \frac{|x|^{\frac{p+1}{p}}}{|\rho|^{\frac{p+1}{p}} - |x|^{\frac{p+1}{p}}}}\, dx = 2P\rho , \qquad (4)$$

and the area of the domain bounded by this curve is

$$A(c_\rho) = 4 \int_{x=0}^{\rho} \left[\rho^{\frac{p+1}{p}} - |x|^{\frac{p+1}{p}} \right] dx = P\rho^2 , \qquad (5)$$

where

$$P = 2 \frac{p}{p+1} B\left(\frac{p}{p+1}, \frac{p}{p+1} \right). \qquad (6)$$

If $p = 1$, then $P = \pi$.

In paper [5] G. D. Chakerian proved and applied the Bonnesen inequality in the Minkowskian plane for any convex *n*-gon (and consequently for any convex curve)

$$L_p \rho \geq A + P\rho^2 . \qquad (7)$$

This inequality was proved by L. Fejes Tóth [8] for nonconvex curves in the Euclidean plane. The proof given in [8] can be generalized without difficulty to such Minkowskian geometry where the “circle” is any centrally symmetric convex curve. The Bonnesen inequality (7) is valid for non-convex curves in Minkowskian geometry [4].

If $p = 1$, inequality (7) is reduced to the Bonnesen inequality valid on the Euclidean plane.

3.1 The isoperimetric inequality in the Minkowskian plane

From the Bonnesen inequality (7) for a simply connected convex domain G. D. Chakerian [5] showed that the isoperimetric inequality in the Minkowskian metric for a simply connected convex domain Ω has the form

$$L_p{}^2 - 4PA \geq 0. \qquad (8)$$

Inequality (8) can be considered as the generalization of the classical isoperimetric inequality (1). In (8) equality holds if and only if domain Ω is bounded by curve c_ρ

$$|x|^{\frac{1}{p}+1} + |y|^{\frac{1}{p}+1} = |\rho|^{\frac{1}{p}+1} .$$

4 Eigenvalue problems

We consider the following eigenvalue problems:

4.1 The linear problem (the problem of a vibrating membrane)

We consider a homogeneous membrane covering a region $\Omega \subset \mathbf{R}^2$. The deformations $u(x, y)$ normal to the plane has to satisfy the differential equation

$$\Delta u + \lambda u = 0 \quad \text{in} \quad \Omega . \qquad (9)$$

On the boundary $\partial\Omega$ of Ω we have the Dirichlet boundary condition:

$$u = 0,$$

if the mebrane is fixed.

The solutions u of problem (9) under Dirichlet boundary condition are called eigenfunctions and the corresponding values of λ are eigenvalues.

In [6] it is showed that there exist countably many number of distinct normalized eigenfunctions with associated eigenvalues to the eigenvalue problem (9). For the eigenvalues $\lambda_j(p)$ of the Dirichlet eigenvalue problem of (9) the relation $\lambda_j(p) \to \infty$ holds when $k \to \infty$. Every eigenvalue is positive.

Only in certain cases, the solutions of (9) can be calculated explicitly. For example, in the case of fixed membranes when Ω is bounded by

- rectangle,
- circle,
- circular segments,
- triangles with angles: $\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}$

$$\frac{\pi}{6},\frac{\pi}{3},\frac{\pi}{2}$$
$$\frac{\pi}{4},\frac{\pi}{4},\frac{\pi}{2}$$

In the case of the linear problem (9) many papers were published on the estimation of the first eigenvalue. Such bounds are based on geometrical data of the domain. The Faber-Krahn inequality (2) gives a lower bound for the smallest (the first) eigenvalue.

For the case of convex domains J. Hersch [9] proved the following bound on the first eigenvalue

$$\lambda_1 \geq \frac{\pi^2}{4\rho^2},$$

where ρ is the radius of the greatest inscribed circle in Ω.

For a simply connected domain Ω, E. Makai [11] showed that there exists a constant C such that

$$\frac{1}{4} \leq C < \frac{\pi^2}{4} \quad \text{and} \quad \lambda_1 \geq \frac{C}{\rho^2}.$$

R. Osserman [12] gave the bound

$$\lambda_1 \geq \begin{cases} \dfrac{1^2}{k\ \rho^2}, & \text{if} \quad k \geq 2, \\ \dfrac{1}{4\rho^2}, & \text{if} \quad k = 1,2 \end{cases}$$

For *k*-fold connected domains.

4.2 The nonlinear problems

We seek eigenfunctions u_j and corresponding eigenvalues $\lambda_j \quad (j = 1,2,...)$ of the following nonlinear eigenvalue problem

$$-Q_p = \lambda |u|^{p-1} u \quad \text{in} \quad \Omega \subset \mathbf{R}^2 \qquad (10)$$

where the nonlinear operator Q_p is defined by

$$Q_p = \frac{\partial}{\partial x}\left(\left|\frac{\partial u}{\partial x}\right|^{p-1} \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y}\left(\left|\frac{\partial u}{\partial y}\right|^{p-1} \frac{\partial u}{\partial y} \right)$$

$$\text{for} \quad 0 < p < \infty.$$

If $p = 1$, problem (10) is equivalent to the linear problem (9).

The boundary condition corresponding to the Dirichlet problem of (10) is

$$u|_{\partial\Omega} = 0.$$

In [3] it is showed that there exist countably many number of distinct normalized eigenfunctions in $W_0^{1,p+1}(\Omega)$ with associated eigenvalues to the eigenvalue problem (10). For the eigenvalues $\lambda_j(p)$ of the Dirichlet eigenvalue problem of (10) the relation $\lambda_j(p) \to \infty$ holds when $k \to \infty$. Every eigenvalue is positive. Here the first eigenfunction has also many special properties. The first eigenfunction does not change sign and the corresponding eigenvalue, the first eigenvalue is simple [13].

We have showed (see [1]) that the Dirichlet problem of (10) has solutions belonging to $C^2(\Omega) \cap C^1(\overline{\Omega})$ when Ω is bounded by rectangle

$$\Omega = \{(x, y) : 0 \leq x \leq a,\ 0 \leq y \leq b\}.$$

For the nonlinear problem with Dirichlet boundary condition the eigenvalues and corresponding eigenfunctions can be given as follows

$$\lambda_{k,l} = p\tilde{\pi}^{p+1}\left(\frac{k^{p+1}}{a^{p+1}} + \frac{l^{p+1}}{b^{p+1}} \right),$$

$$u_{k,l} = A_{k,l} S_p\left(\frac{k\tilde{\pi}}{a} x \right) S_p\left(\frac{l\tilde{\pi}}{b} y \right),$$

$$k, l = 1,2,...,$$

where

$$\tilde{\pi} = \frac{2\dfrac{\pi}{p+1}}{\sin\dfrac{\pi}{p+1}},$$

$A_{k,l} = const.$ determined from the normalization of the eigenfunction, and function S_p is the solution of the differential equation

$$S_p'' |S_p'|^{p-1} + |S_p|^{p-1} S_p = 0$$

under condition

$$S_p(0) = 0, \quad S_p(\tilde{\pi}) = 0.$$

The function S_p is the generalized sine function, which plays the same role in case of nonlinear problem (10) as the sine function in case of the linear problem (9).

Isoperimetric inequalities are also useful in the derivation of explicit a priori inequalities employed in the determination of a priori bounds in various types of initial or boundary value problems. As an example, we know that for domain Ω with sufficiently smooth boundary $\partial\Omega$ the first eigenvalue in the fixed membrane problem (with boundary condition $u|_{\partial\Omega} = 0$) admits the following characterization:

$$\lambda_1 = \min_{u \in W_0^{1,p+1}(\Omega)} \frac{\iint_\Omega \left(\left| \frac{du}{dx} \right|^{p+1} + \left| \frac{du}{dy} \right|^{p+1} \right) dx\, dy}{\iint_\Omega |u|^{p+1} dx\, dy}.$$

This characterization gives us a bound for λ_1, i.e., for any $v \in W_0^{1,p+1}(\Omega)$

$$\lambda_1 \le \frac{\iint_\Omega \left(\left| \frac{dv}{dx} \right|^{p+1} + \left| \frac{dv}{dy} \right|^{p+1} \right) dx\, dy}{\iint_\Omega |v|^{p+1} dx\, dy}.$$

The equality sign will always hold for some choice of v

In [4] we gave a lower bound for the first eigenvalue of the nonlinear eigenvalue problem (10). By using geometrical data we get

$$\lambda_1(p) \ge \left(\frac{P h_0^{\;2}}{A} \right)^{\frac{p+1}{2}} \tag{11}$$

where A is the area of $\Omega \subset \mathbf{R}^2$, P is defined in (6), and h_0 is the first positive zero of the generalized nonlinear Bessel function $H_0(x)$ satisfying the nonlinear ordinary differential equation (see[2])

$$\frac{d}{dx}\left[\left| \frac{dH_0}{dx} \right|^{p-1} \frac{dH_0}{dx} \right] + \frac{1}{x} \left| \frac{dH_0}{dx} \right|^{p-1} \frac{dH_0}{dx} + $$
$$+ \lambda |H_0|^{p-1} H_0 = 0$$

with conditions

$$H_0(0) = 1$$

and

$$\frac{dH_0}{dx}(0) = 0.$$

In (11) equality holds if and only if domain Ω is bounded by curve c_ρ. Inequality (11) is a generalization of the Faber-Krahn inequality for nonlinear eigenvalue problems.

Another lower bound can be given for the first eigenvalue of the nonlinear problem (10) by using the method of Steiner symmetrization. We know that the eigenfunction associated to λ_1 has the same sign in Ω. If domain Ω is a simply connected convex domain in $\mathbf{R}^2$, then

$$\lambda_1 \ge \left[\frac{A + \sigma}{(p+1)\rho A} \right]^{p+1} \tag{12}$$

where A is the area of $\Omega \subset \mathbf{R}^2$, ρ is the radius of the greatest inscribed curve c_ρ of Ω, and $\sigma = P\rho^2$ is the area of the region bounded by the greatest inscribed curve c_ρ of Ω.

In (12) equality holds if and only if domain $\Omega \subset \mathbf{R}^2$ is bounded by curve c_ρ. Inequality (12) is the generalization of the estimation given by E. Makai for the linear eigenvalue problem (9) in [11].

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