A Model for a Turbulence System Generator

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Abstract: - In this paper we propose a model of a nonautonomous dynamical system whose operations show turbulent behavior. The appearance of the trajectory is similar to that of a simplified tornado. The electronic circuit implementation of this model contains simply two linear capacitors, a voltage-controlled switch (VCS), a linear inductor, two resistors, and a nonlinear negative resistor. Computer simulation results are used to show the existence of the desired turbulent attractor.

Key-Words: - Turbulence modelling, chaos, attractor, bifurcation, oscillator

1. Introduction

Turbulence is a nonlinear phenomenon that can be found in many fields of science and technology. It has received considerable attention in several developments of high-speed jet aircraft, plasma physics, and chemical engineering [1]. Thus, the development of a model of a simple system whose behavior is turbulent, is important not only in the study of how to avoid this nonlinear phenomena, which can be a nuisance, but also in the development of applications which actually exploit the features of this phenomenon. Brown [2] has discussed the relationship between turbulence and chaos and provided the theory to synthesize a turbulent vortex with convection. Inspired by this theory and the design of the twist-and flip circuit [3,4], we can actually develop a model of a system with a turbulent behavior in a same way as designing chaotic systems.

In this paper, we propose a model for a turbulence system generator whose phase portrait resembles that of a simplified tornado, i.e. a rotating trajectory having a vortex and swirling upward and downward. In section 2, an nonautonomous system design of this model is presented. This system is constructed from building blocks (e.g. integrator, summer, multiplier, and amplifiers) that can be either designed or found in the off-shelf electronics library. The dynamics of this system, which is driven by a square-wave voltage source, are described by a system of ordinary differential equations

$$\dot{X} = f(X,t) \tag{1}$$

Where X is a vector in 3-dimensional Euclidean space R^3. In addition, a step-by-step procedure on how to select the system parameters is also presented. Section 3 presents a simple nonautonomous electronic circuit implementation with a turbulent behavior. This circuit contains only two capacitors, one inductor, two resistors, a voltage-controlled switch (VCS) and a nonlinear negative resistor. Because of its simplicity, this circuit can be a good starting point to design more complex turbulent systems for advanced applications. Enhancements of the proposed system are also discussed.

Throughout the paper, computer simulation results, using Simulink [5] and Electronics Workbench [6], are obtained to show the existence of the desired turbulent attractor.

2. Turbulent system design

Our aim in this section is to develop a model of a nonautonomous system that exhibits a trajectory that resembles that of a simplified tornado. It should be clear that the proposed model is kept very simple for illustration purpose and is not meant to represent actual tornados. Fig.1(a) shows the basic system model.

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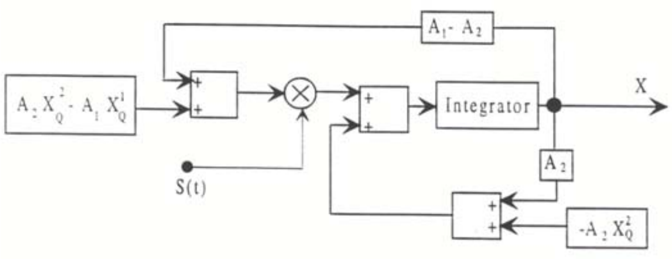

Fig.1(a). Block diagram of a turbulent system.

This model is described by the following system of ordinary differential equations:

$$\dot{X}(t) = s(t)A_1\left(X(t) - X_Q^1\right) + \left(1 - s(t)\right)A_2\left(X(t) - X_Q^2\right) \quad (2)$$

Where,

$$A_1 = \begin{bmatrix} \alpha_1 & -\omega_1 & 0 \\ \omega_1 & \alpha_1 & 0 \\ 0 & 0 & \gamma_1 \end{bmatrix}, \; A_2 = \begin{bmatrix} \alpha_2 & -\omega_2 & 0 \\ \omega_2 & \alpha_2 & 0 \\ 0 & 0 & \gamma_2 \end{bmatrix}$$

$$X_Q^1 = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}, \; X_Q^2 = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}, \; and \;\; X(0) = X_0 = \begin{pmatrix} x(0) \\ y(0) \\ z(0) \end{pmatrix}$$

$X(0)$ is the initial condition, and $\dot{X}(t)$ denotes the derivative of the trajectory X(t). X_Q^1 and X_Q^2 are the equilibrium points. The voltage source s(t), for driving the system, is selected to be a square wave of unity amplitude and angular frequency f or period $\left(T = \dfrac{1}{f}\right)$ as shown in Fig.1(b).

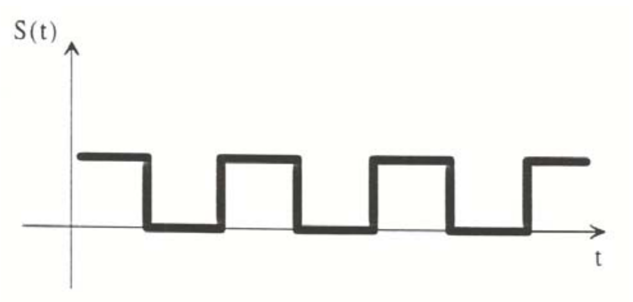

Fig.1(b). Waveform of the voltage source s(t) (a square wave with unity amplitude and frequency f.)

It is described by:

$$s(t) = \begin{cases} 1 & for \quad t \in (n, n + \frac{1}{2})T \\ 0 & for \quad t \in (n + \frac{1}{2}, n + 1)T \end{cases} \quad (3)$$

Where n=0,1,2….For s(t)=1, the trajectory of system (2) is lead by the eigenvalues $(\alpha_1 \pm j\omega_1, \gamma_1)$ and eigenvectors of A_1 and the location of the equilibrium point X_Q^1. For s(t)=0, it is lead by the eigenvalues $(\alpha_2 \pm j\omega_2, \gamma_2)$ and eigenvectors of A_2 and the location of the equilibrium point X_Q^2. Equation (2) can be regarded as three linear equations on two half-spaces

$$\begin{aligned} S_1 &= \left\{(x, y, z), such\ that\ s(t) = 1\right\} \\ S_0 &= \left\{(x, y, z), such\ that\ s(t) = 0\right\} \end{aligned}$$

The solution of (2) on S_1 is given by

$$X(t) = X_Q^1 + 2c_{c_1} e^{(\alpha_1 t)}\left[\cos\left(\omega_1 t + \varphi_{c_1}\right)\vec{\eta}_{r_1} - \sin\left(\omega_1 t + \varphi_{c_1}\right)\vec{\eta}_{i_1}\right] + c_{r_1} e^{(\gamma_1 t)}\vec{\xi}_{\gamma_1} \quad (4)$$

where $\vec{\eta}_{r_1}$ and $\vec{\eta}_{i_1}$ are the real and imaginary parts of the eigenvectors associated with the complex conjugate pair of eigenvalues, $\vec{\xi}_{\gamma_1}$ is the eigenvector associated with the real eigenvalue. The constants c_{c_1}, c_{r_1}, and φ_{c_1} are real and determined by the initial conditions X(0). Similarly, the solution of (2) on S_0 can be written in the same form as (4) with the corresponding eigenvalues, eigenvectors and constants. The objective is then to select all of the systems parameters described by (2) such that the trajectory of this system swirls down as it approaches the equilibrium point X_Q^1. Once it reaches the bottom, it changes directions and swirls up to the top away from the equilibrium point X_Q^2. This process is repeated over again. To achieve this objective, a step-by-step procedure is presented next. It should be noted that the selection of the parameters in this procedure is not unique. The function s(t) is selected to be periodic for simplicity and it should be clear that this does not imply that the trajectory of an actual tornado is periodic.

Design Procedure

Step 1: *Design of top-to-bottom trajectory*
Specifications 1: For the dynamical system described by A_1 to have a trajectory that swirls down around X_Q^1 while approaching it in the (x,y) plane and repelling it in the z-direction, the following must be satisfied :

(i) The complex eigenvalues $\lambda = \alpha_1 \pm j\omega_1$ must be selected such that $\alpha_1<0$ and $\omega_1>0$.
(ii) The real eigenvalue $\gamma_1>0$.

Step 2: *Design of bottom-to-top trajectory*
Specifications 2: For the dynamical system described by A_2 to have a trajectory that swirls up around X_Q^2 while repelling it in the (x,y) plane and approaching it in the z-direction , the following must be satisfied :

(i) The complex eigenvalues $\lambda = \alpha_2 \pm j\omega_2$ must be selected such that $\alpha_2>0$ and $\omega_2>0$.
(ii) The real eigenvalue $\gamma_2<0$.

Step 3: *Linking the two trajectories*
In this step, select the function s(t) and use equation (2) to link the systems described in steps (1) and (2). Note that the initial conditions $X(0)$ and the equilibrium points X_Q^1 and X_Q^2 must be selected to satisfy the above specifications.

This procedure is best illustrated by the following example.

Example 1: Consider the following dynamical system described by equation (2) where,

$$A_1 = \begin{bmatrix} -0.1 & -4.0 & 0 \\ 4.0 & -0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix},\ A_2 = \begin{bmatrix} 0.2 & -5.0 & 0 \\ 5.0 & 0.2 & 0 \\ 0 & 0 & -0.2 \end{bmatrix}$$

$$X_Q^1 = \begin{pmatrix} 1.0 \\ 1.0 \\ 2.0 \end{pmatrix}, X_Q^2 = \begin{pmatrix} 1.1 \\ 1.1 \\ 2.1 \end{pmatrix}, X(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}, X(0) = \begin{pmatrix} 2.0 \\ 2.0 \\ 1.0 \end{pmatrix}$$

The implementation of this system is shown in Fig.2. Fig.3 shows a trajectory that swirls around in the direction toward X_Q^1 and away from X_Q^2. This trajectory meets the design specifications. Starting from the initial state $X(0)$, and for s(t)=1, it swirls down until s(t) changes to 0. At that time, the trajectory will swirls up until s(t) changes back to 1. The process repeats itself.

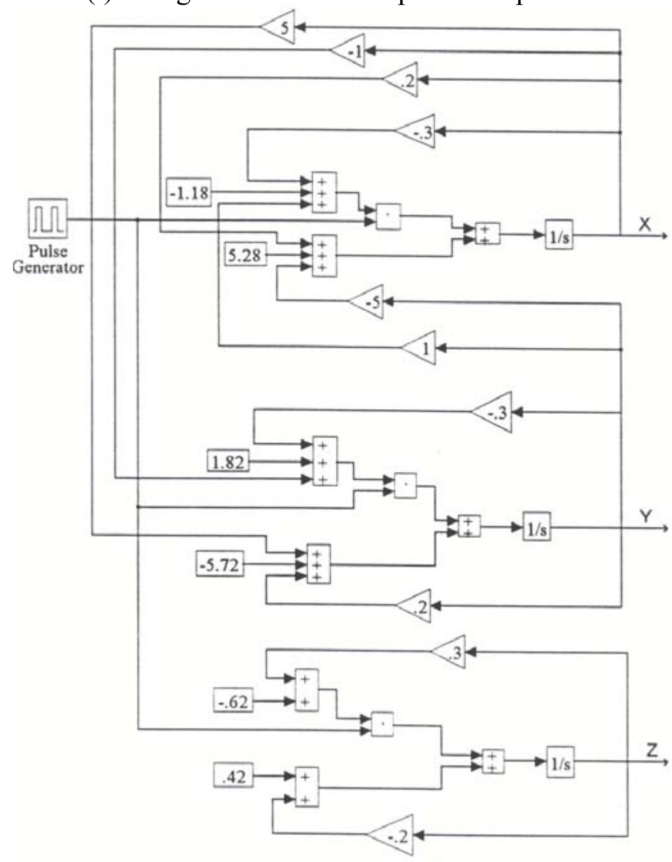

Fig.2 Block diagram of the physical implementation of the turbulent system described in the example.

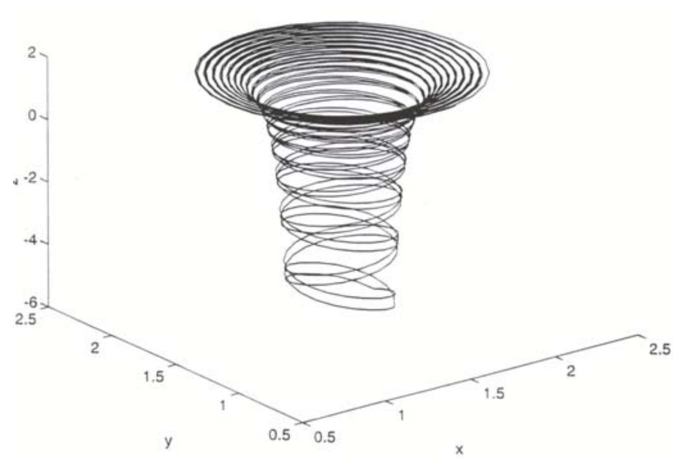

Fig.3. Trajectory of the system of Fig.2 realizing the dynamics of turbulence similar to that of a simplified tornado.

In the next section, a simple electronic circuit that displays a turbulent behavior is presented. The design of this electronic circuit is important not only to understand further the nature of turbulence but also to serve as a starting point to develop applications that explore this nonlinear behavior.

3. Turbulent circuit implementation

Fig. 4 shows an electronic circuit implementation of the model described in section 2. This circuit is simple and can be viewed as a good starting point to design complex systems that exploit the phenomenon of turbulence.

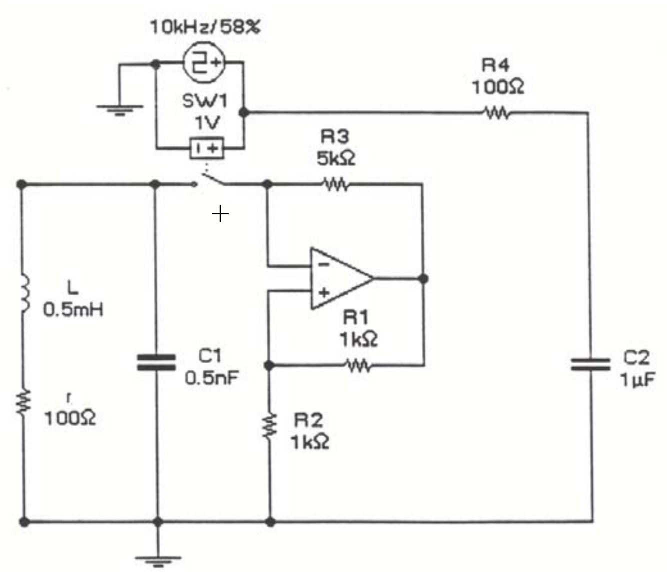

Fig. 4. Implementation of the turbulent circuit

The proposed circuit contains simply two linear capacitors $C_1$and C_2, a voltage controlled switch (VCS) SW_1, a linear inductor L, two resistors r and R_4, and a nonlinear negative resistor. The nonlinear negative resistor denoted by R, described by *i(v),* is implemented using an operational amplifier and three resistors R_1, R_2, and R_3. (v is the voltage between the negative terminal of the opamp and the ground).

$$i(v) = \begin{cases} \dfrac{1}{R_3}(v - V_{sat}) & if \quad v < \beta V_{sat} \\ -\dfrac{R_1}{R_2 R_3} v & if \quad -\beta V_{sat} < v < \beta V_{sat} \\ \dfrac{1}{R_3}(v + V_{sat}) & if \quad v > -\beta V_{sat} \end{cases} \qquad (5)$$

Where $\beta = R_2 / (R_1 + R_2)$. The case of interest is when this part of the circuit is active (i.e. generating energy for the capacitor and inductor). In this case, $R = \left(-R_1 / R_2 R_3\right)^{-1} \; for \; -\beta V_{sat} < v < \beta V_{sat}$. V_{sat} is the operational amplifier saturation voltage. The switch SW_1 is controlled by the voltage source s(t) which is a square wave of unity amplitude, frequency f and duty cycle d. The switch SW_1 is closed when s(t)=1, otherwise it is open. The state equations of the turbulent circuit are described for both cases, s(t)=1 and s(t)=0:

For s(t)=1, we have $v = v_{c1}$ and

$$\begin{bmatrix} C_1 \dot{v}_{c1} \\ L \dot{i}_L \\ C_2 \dot{v}_{c2} \end{bmatrix} = \begin{bmatrix} R & -1 & 0 \\ 1 & -r & 0 \\ 0 & 0 & -1/R_4 \end{bmatrix} \begin{bmatrix} v_{c1} \\ i_L \\ v_{c2} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/R_4 \end{bmatrix} \qquad (6)$$

and for s(t) = 0, we have

$$\begin{bmatrix} C_1 \dot{v}_{c1} \\ L \dot{i}_L \\ C_2 \dot{v}_{c2} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & -r & 0 \\ 0 & 0 & 1/R_4 \end{bmatrix} \begin{bmatrix} v_{c1} \\ i_L \\ v_{c2} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (7)$$

In accordance with the previous design considerations, suitable values for the circuit components are: L=0.5mH, r=100Ω, C_1=0.5nF, C_2=1µF, R_1=R_2=1kΩ, R_3=5kΩ, and R_4=100Ω. The voltage source s(t) has a frequency f=10kHz and a duty cycle d=58%. The equilibrium points are located at (0,0,-1) and (0,0,0) and the initial conditions are selected to be ($v_{c1}(0)$=0.2V, $i_L(0)$=0, $v_{c2}(0)$=0). The phase portraits in the planes v_1-v_2 and i_L-v_1, obtained by simulating the circuit using Electronics Workbench [6], are shown in Fig.5 and Fig.6, respectively.

It should be clear that although the model described above is kept simple, adding several features could enhance it. For instance, we can include the following feature such as (1) increasing the trajectory rotation speed and (2) moving the fixed points. These enhancements can be related to the atmosphere changes.

4. Conclusion

In this paper, we have proposed a model of a nonautonomous dynamical system that shows turbulent behavior similar to that of a simplified tornado. The proposed system is constructed from simple building blocks that can be easily described by linear ordinary differential equations. The global system is driven by an external source that enables the trajectory to swirl up and down. A simple electronic circuit implementation displaying turbulence was also presented. Although the true dynamics of turbulence is certainly much more complex, the proposed simplified model may form the basis for the development of more advanced and detailed mathematical models. It may also serve as a starting point for further research to understand this complex phenomenon of turbulence and to develop applications that use actually this phenomenon.

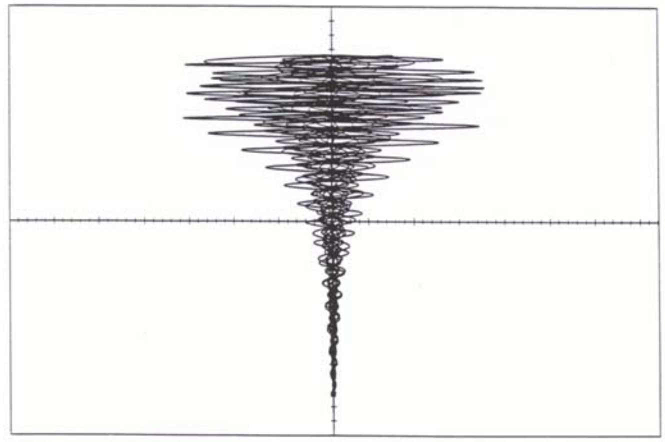

Fig. 5. Phase portrait in the plane v_1-v_2. (horizontal axis: v_1, 5V/div. vertical axis: v_2, 0.1V/div).

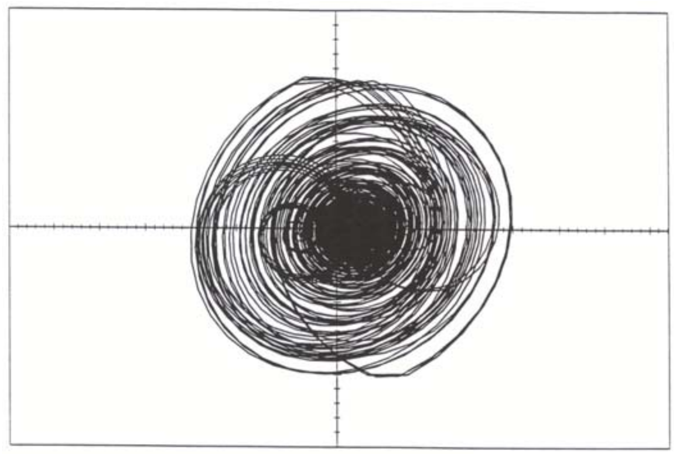

Fig. 6. Phase portrait in the plane i_L-v_1 (horizontal axis: i_L, 0.1V/div. vertical axis: v_1, 5V/div).

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